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Flow of Suspensions through Porous Media—

New differential equation for clogged beds is derived

The flow of suspensions through porous media is a very complex phenomenon owing to the diversity of the mechanisms involved. However, the study of these mechanisms shows that it is possible to define two types of deep filtration—a mechanical filtration for large particles (diameter $\geqslant$ 30 μ), and a physicochemical filtration for small particles (diameter about 1 μ)—for mean particles (3 μ < diameter < 30 μ), both mechanical and physicochemical phenomena intervene. The clogging (colmatage) of a porous medium is described mathematically by the mass balance equation and the kinetic equations describing the rate of clogging and the rate of decolmatage. Generally, the mass balance equation simplifies since diffusion of particles is not important nor are suspended particles with respect to retained ones. Usually, the kinetic equation is assumed to be of first order and decolmatage of previously retained particles is considered to be negligible as is usually found experimentally. With these assumptions, a new differential equation for clogged beds is derived. It is valid for all relationships between the retention probabilities and the retention. From this equation, general macroscopic properties are deduced.

However, it is difficult accurately to connect the experimentally determined macroscopic parameters with the elementary mechanisms of clogging—retention sites, retention forces, and rate of elementary capture processes. Likewise, it is difficult to express the variation in the pressure drop of the fluid through the clogged bed as a function of the retention—A number of improvements of the Kozeny model have been tried, with limited success.

PROBLEM STATEMENT

Liquid-solid separation by filtration can be accomplished either by solid accumulation in front of a filter medium, or by retention inside a deep porous bed. In the first case, a filter cake is built-up and it stops the suspended particles. In the second case, the suspension flows through the porous medium in which the particles are retained. The flow of suspensions through porous media is quite a natural phenomenon since it often occurs when water seeps into geological masses. The use of deep filtration in the treatment of polluted water is not new, but this phenomenon has only been studied recently.

When flowing through porous medium, the particles are brought into contact with the possible retention sites; they stop there or are carried away by the stream. The deep filtration is therefore the result of several mechanisms:

the contacting of particles with the retention sites
the fixing of particles on sites

and eventually,

the breaking away of previously retained particles

Thus, the problem to be solved consists in relating the clogging rate—*i.e.*, the number of retained particles per unit of time and per unit of volume of porous medium—to the various factors which define the system, namely

the carrier fluid (flowrate, viscosity, density)
the suspension of particles (concentration, size, and shape of particles)

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Application to Deep Filtration

the porous medium filter (porosity, diameter of pores, size and shape of grains, retention—*i.e.*, volume fraction already clogged)

Furthermore, the important problem of the pressure drop of the fluid through the bed has to be studied as well as its change with the clogging. To approach this problem rationally, the authors will first establish the mass balance equations for a medium in the process of clogging. With reasonable assumptions, these equations are simplified to allow convenient comparison with experiments.

In a second part, the elementary mechanisms which cause the retention of particles in a porous medium are identified. From the basic laws of particles mechanics and the physical chemistry of interfaces, the authors derive the general form of the equations governing the clogging "declogging" phenomena.

The third part combines these equations and the mass balance equations to describe the general properties of these systems. The conditions in which the complete integration of these differential equations is possible are also discussed.

The fourth part examines the pressure drop and its variation with clogging.

MASS BALANCE EQUATIONS

Exact Equations

Mass balance equations for the particles and the carrier fluid.

Particles balance. Consider a porous medium element of depth Δz and area Ω (Figure 1), which contains:

a volume $\Omega\sigma\Delta z$ of retained particles
a volume $\Omega\epsilon y\Delta z$ of moving particles entrained by the liquid

where

σ is the retention—*i.e.*, the volume of deposited particles per unit of filter volume
y is the volume fraction of particles in the suspension—*i.e.*, the volume of particles per unit of suspension volume
ϵ is the porosity of the bed—*i.e.*, the volume occupied by the flowing suspension per unit of filter volume

But the retained particles imprison between themselves some liquid called "dead liquid" and form a deposit of a real volume $\Omega\beta\sigma\Delta z$. β is the inverse of the

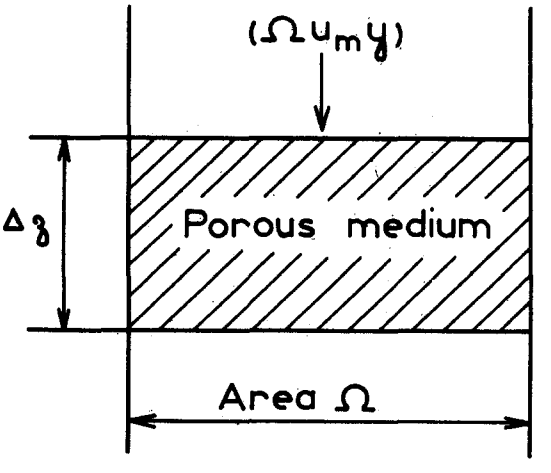


Figure 1. Porous medium element

compaction factor of the retained matter. The porosity of the clogged bed can be expressed by the relation

$$\epsilon = \epsilon_o - \beta\sigma \tag{1}$$

where ϵ_o is the porosity of the clean bed.

The general mass balance equation is

$$\text{accumulation rate} = (\text{flow in}) - (\text{flow out})$$

or $$\frac{\partial}{\partial t}[\Omega(\sigma + \epsilon y)] + \frac{\partial}{\partial z}\left[\Omega u_m y - \Omega D \frac{\partial y}{\partial z}\right] = 0$$

where

u_m = approach velocity of the suspension
$u_m y$ = barycentric flow of particles entrained by the fluid
$-D\frac{\partial y}{\partial z}$ = diffusional flux of particles

Usually the suspension flow rate is kept constant in deep filtration and the exact form of the mass balance for the particles is

$$\frac{\partial}{\partial t}(\sigma + \epsilon y) + u_m \frac{\partial y}{\partial z} - D\frac{\partial^2 y}{\partial z^2} = 0 \tag{2}$$

Liquid balance equation. The same porous medium element contains a volume $\Omega(\beta - 1)\sigma\Delta z$ of dead liquid and a volume $\Omega\epsilon(1 - y)\Delta z$ of flowing liquid.

Thus, the liquid balance equation is

$$\frac{\partial}{\partial t}[\Omega\epsilon(1 - y) + \Omega(\beta - 1)\sigma] + \frac{\partial}{\partial z}\left[\Omega u_m(1 - y) + \Omega D\frac{\partial y}{\partial z}\right] = 0$$

The term $u_m(1 - y)$ describes the barycentric flow of liquid, and $+D(\partial y/\partial z)$ represents the diffusional flux opposed to that of the particles, since the porous bed is always filled with suspension.

If the flow rate is kept constant

$$\frac{\partial}{\partial t}\epsilon(1 - y) + \frac{\partial}{\partial t}(\beta - 1)\sigma + u_m\frac{\partial}{\partial z}(1 - y) + D\frac{\partial^2 y}{\partial z^2} = 0 \tag{3}$$

which is the exact balance equation for the liquid.

With Equation 1 between ϵ and ϵ_o taken into account, Equation 3 becomes

$$\frac{\partial \epsilon_o}{\partial t} - \frac{\partial}{\partial t}(\sigma + \epsilon y) - u_m\frac{\partial y}{\partial z} + D\frac{\partial^2 y}{\partial z^2} = 0$$

or since ϵ_o is a constant

$$\frac{\partial}{\partial t}(\sigma + \epsilon y) + u_m\frac{\partial y}{\partial z} - D\frac{\partial^2 y}{\partial z^2} = 0$$

which is identical with particles balance Equation 2. This is not surprising because the porous medium is fully filled with suspension, and every time a particle stops, it drives away a volume of carrier liquid equal to its own volume, and consequently, it is normal that the balance equations are the same for the particles and the liquid. The mass balance in the porous bed is characterized completely by Equation 2.

Approximate Equations

Three simplification modes for exact Equation 2 are considered.

Third approximation. Particles diffusion is negligible when their size is larger than 1 μ; even for ultrafine particles, the barycentric flow is always the most significant; thus, the diffusional term may be neglected. This case is denoted "third approximation" which is characterized by Equation 4:

$$\frac{\partial(\sigma + \epsilon y)}{\partial t} + u_m\frac{\partial y}{\partial z} = 0 \tag{4}$$

This assumption is usually made; only a few investigations take into consideration particles diffusion (*38*, *45*, *44*, *48*). Even with this simplification, the balance equation remains complicated and is little used under this form (*22*, *63*) because it is usually further simplified by additional assumptions.

Second approximation. The term $\beta\sigma$ is often neglected with respect to ϵ_o in the balance equation (*3*, *48*, *71*, *72*) and with $\epsilon_o \simeq \epsilon$, the balance equation becomes

$$\frac{\partial\sigma}{\partial t} + \epsilon_o\frac{\partial y}{\partial t} + u_m\frac{\partial y}{\partial z} = 0 \tag{5}$$

This simplification, called the "second approximation," consists of neglecting $\beta\sigma y$ with respect to $\sigma + \epsilon_o y$. It is justified only when clogging starts, when σ is small and ϵ close to ϵ_o, but is always valid for deep filtration since concentrations are low. The error is

$$\frac{\beta\sigma y}{\sigma + \epsilon_o y} = \frac{\beta y}{1 + \epsilon_o y/\sigma}$$

and always remains lower than βy for all values of σ. Generally, concentrations are about 0.1% and β may be estimated at 2 or 3; thus, the maximum error is 0.3% and therefore negligible.

First approximation. If, as it is often assumed, all moving particles are neglected with respect to the retained particles (*22*, *24*, *28*, *35*, *54*, *57*, *70*), the balance equation is

$$\frac{\partial\sigma}{\partial t} + u_m\frac{\partial y}{\partial z} = 0 \tag{6}$$

This simplification, called the "first approximation," is only reasonable if ϵy is small compared with σ. Since

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ϵy is always less than $\epsilon_o y_i$ (y_i is the concentration of the suspension applied to the filter) the error is at most $\epsilon_o y_i/\sigma$; thus the simplification is best for large values of σ.

For example, if $\epsilon_o = 0.40$ and $y_i = 0.1\%$ the error is 8% at most for $\sigma = 0.50\%$ and only 1% for $\sigma = 4.0\%$.

Comparison of first and second approximations. The second approximation is justified for the duration of the filtration, whereas the first is only valid for σ sufficiently large and is invalid when clogging starts. Consequently, it seems logical to use the first as soon as the required conditions are satisfied (*i.e.*, when σ is large enough) and the second for the beginning of clogging. These two approximations are complementary, and it is even possible to group them under the same equation.

Retention age. Generally, before the suspension enters the porous bed, the latter is filled with clear liquid which is driven away by the suspension. If the bed is deep, the suspension needs a substantial time to pass through it; accordingly, the instant when the suspension reaches a bed, element is not the same all along the porous medium. Thus, it is necessary to compare the bed elements at the same "retention age"—*i.e.*, at different instants t such as $t - \epsilon_o z/u_m = \text{constant}$.

The term $\epsilon_o z/u_m$ is the required time so the suspension arrives at the bed depth, z.

The "retention age," τ, is defined by

$$\tau = t - \frac{\epsilon_o z}{u_m} \tag{7}$$

The term $\epsilon_o z/u_m$ is substantial when clogging starts—*e.g.*, if $u_m = 0.1$ cm/sec and $\epsilon_o = 0.40$, the suspension needs 200 sec to reach the depth of 50 cm; but as deep filtration operations normally last several hours this difference becomes negligible fairly quickly and for most of the filtration, t and τ may be identified.

To follow a suspension element when it passes through the clean bed, it is necessary to keep $\tau = \text{constant}$. This is not exact for a clogged bed because the porosity and therefore, the suspension velocity are changed. At constant flow rate, it is necessary to choose instants t such as

$$t - \int_0^z \frac{\epsilon z}{u_m}\, dz = \text{constant}$$

or

$$\tau + \int_0^z \frac{\beta\sigma}{u_m}\, dz = \text{constant}$$

Now the term

$$\int_0^z \frac{\beta\sigma}{u_m}\, dz$$

is a correction term that occurs only when σ is large enough—*i.e.*, when the filtration has lasted for some time. The difference between t and τ is now negligible; consequently, this correction is unnecessary and even for a clogged bed, a suspension element passes through the bed at $\tau = \text{constant}$.

Transformation of balance Equation 5. Substituting $t = \tau + \epsilon_o z/u_m$ for t in balance Equation 5 of the second approximation yields Equation 5′

$$\frac{\partial\sigma}{\partial\tau} + u_m \frac{\partial y}{\partial z} = 0 \tag{5'}$$

Equations 5′ and 6 have an identical mathematical form but in the former, the variables are τ and z and in Equation 6, t and z. This may be explained as follows: writing the balance equation with the variable τ consists of following a suspension element that passes through the bed; for this element, the decrease in concentration is equal to the increase in bed retention (Equation 5′). On the other hand, writing the balance equation with the variable t consists in taking a fixed-bed element; the suspension which passes through it may have a concentration changing with the time (term $\partial\epsilon y/\partial t$ in Equation 2). In the first approximation, these changes in concentration with the time are neglected, and the filtration is treated as if the decrease in concentration was only due to clogging. Thus, it is expected that Equations 5′ and 6 have the same mathematical form.

CLOGGING MECHANISMS—THE KINETIC EQUATION

Although the mass balance equation is independent of the clogging mechanisms, the kinetic equation, which describes the rate of the transfer of particles to the porous medium, is a function of the elementary processes of this transfer. Therefore, it cannot be obtained in a strictly theoretical way as the mass balance equation and must be inferred from experiments.

Retention probabilities definition. Consider again the porous bed element of area Ω and depth Δz (Figure 1); during the small time interval Δt, the retention $\Omega\sigma\Delta z$ increases to $\Omega(\sigma + (\partial\sigma/\partial t)\,\Delta t)\Delta z$ and a volume $\Omega u_m y \Delta t$ of suspended particles enters. The retention probability q of a particle in this element will be defined by

$$q = \frac{\Omega\Delta z \dfrac{\partial\sigma}{\partial t}\,\Delta t}{\Omega u_m y\,\Delta t} = \frac{\Delta z}{u_m y}\frac{\partial\sigma}{\partial t}$$

or by dividing by Δz the retention probability per unit of depth, k, is obtained:

$$k = \frac{1}{u_m y}\frac{\partial\sigma}{\partial t} \tag{8}$$

It is also possible to define a retention probability, k', of a particle per unit time: if a particle has a probability k to be retained in the unit bed depth, then it has the probability k to be retained during the time interval required to pass through a layer of unit depth; the average time interval for passage is ϵ/u_m. Thus

$$k' = k\frac{u_m}{\epsilon}$$

$$k' = \frac{1}{\epsilon y}\frac{\partial\sigma}{\partial t} \tag{9}$$

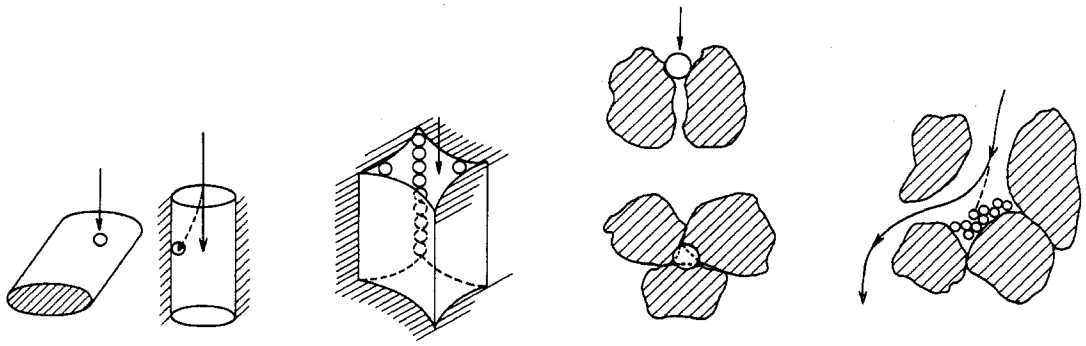


Figure 2. Retention sites

Such a definition of retention probabilities does not distinguish the two directions of the particles exchange:

suspension → porous medium (colmatage or clogging or capture)

porous medium → suspension (decolmatage or scouring)

Definition of the colmatage (capture) and decolmatage (de-clogging) probabilities. If in the porous bed element colmatage and decolmatage are simultaneous, the variation in the retention per unit time $\Omega\Delta z(\partial\sigma/\partial t)$ can be considered as the sum of the increase due to the capture of particles, indicated by $\Omega\Delta z(\partial\sigma/\partial t)_c$ and of the decrease owing to the decolmatage, indicated by $\Omega\Delta z(\partial\sigma/\partial t)_r$.

Thus, the probability of capture of a particle per unit time will be defined by

$$k_c' = \frac{1}{\epsilon y}\left(\frac{\partial\sigma}{\partial t}\right)_c \tag{10}$$

and the probability of decolmatage of a particle per unit time will be

$$k_r' = -\frac{\Omega\Delta z\left(\frac{\partial\sigma}{\partial t}\right)_r}{\Omega\sigma\Delta z}$$

or

$$k_r' = -\frac{1}{\sigma}\left(\frac{\partial\sigma}{\partial t}\right)_r \tag{11}$$

Under these conditions, the retention probability per unit time k' will be written

$$k' = k_c' - \frac{k_r'\sigma}{\epsilon y} \tag{12}$$

Description of Elementary Mechanisms

Consider the inventory of all parameters which describe the elementary processes of clogging and decolmatage—*i.e.*, the retention sites, the retention forces exerted on the particles retained in these sites, the capture mechanisms which bring the particles into contact with the sites, as well as the decolmatage processes of retained particles.

Retention sites. It is possible to distinguish several retention sites (Figure 2):

Surface sites. The particle stops and is retained on the surface of a porous bed grain.

Crevice sites. The particle becomes wedged between the two convex surfaces of two grains.

Constriction sites. The particle cannot penetrate into a pore of a smaller size than its own.

Cavern sites. The particle is retained in a sheltered area, a small pocket formed by several grains.

Retention forces. They include:

Axial pressure of the fluid. The fluid pressure may hold an immobilized particle against the opening at a constriction.

Friction forces. A particle wedged in a crevice may have been slightly deformed when stopped, and may remain in place by friction.

Surface forces. These include the Van der Waals forces, which are always attractive, and the electrical forces (electrostatic or electrokinetic) which are either attractive or repulsive according to the physicochemical conditions of the suspension.

Chemical forces. In the case of colloidal particles or in other particular cases, actual chemical bonding may occur.

Capture processes. These are:

Sedimentation. If the particles have a density different from that of the liquid, they are subjected to gravity and their velocity no longer is that of the fluid; thus, by sedimentation they can meet the filter medium.

Inertia. Still owing to their apparent weight, the particles cannot follow the same trajectories as the fluid, they deviate from the streamlines (when the directions of the trajectories change suddenly) and can be brought into contact with the bed grains.

Hydrodynamic effects. Owing to the nonuniform shear field and the nonsphericity of particles, hydrodynamic effects may occur; these effects cause a lateral migration of suspended particles which may be brought into contact in this way with retention sites.

Direct interception. Even with exactly the same density as the fluid, the particles would not be able, owing to their size, to follow the smallest tortuosities of the streamlines of the carrier fluid and they will thus collide with the walls of the convergent areas of the pores.

Diffusion by Brownian motion. The particles diffuse and can reach areas which are not normally irrigated by the suspension, and they are retained there.

Decolmatage processes. It is necessary to distinguish between the spontaneous decolmatage due to the normal flow of suspension through the clogged bed, and the decolmatage caused by the operator who suddenly changes the flow conditions.

Table I. Ratio of Inertial to Gravity Forces ($\epsilon = 0.40$)

$u_m : d_g$ Cm/sec	100 μ	500 μ	1000 μ	1500 μ
0.001	1.3×10^{-6}	2.6×10^{-7}	1.3×10^{-7}	0.85×10^{-7}
0.010	1.3×10^{-4}	2.6×10^{-5}	1.3×10^{-5}	0.85×10^{-5}
0.100	1.3×10^{-2}	2.6×10^{-3}	1.3×10^{-3}	0.85×10^{-3}
1.0	1.3	0.26	0.13	0.085

Spontaneous decolmatage may occur if local variations in pressure or flow rate change the flow in the neighborhood of retained particles or if a moving particle collides with a retained particle.

Provoked decolmatage results from impulses—*i.e.*, from sudden variations in pressure or flow rate in the whole bed caused by the operator or by reversal of the flow direction.

The processes of spontaneous or provoked decolmatage are similar but the extent of the first is local, whereas the second occurs everywhere in the whole bed.

Factors acting upon clogging kinetics. Consider the experimental parameters which are easily varied and which have an effect upon clogging kinetics:

The carrier fluid is characterized by its dynamic viscosity η_l, its specific mass ρ_l, and its flow rate.

The suspension is defined by the concentration y of particles which are assumed spherical (of mean diameter d), their specific mass is ρ_S.

The filter porous medium is characterized at the initial instant (designed by the index $_o$) by its permeability B_o (Darcy's Law) and its porosity ϵ_o. The bed is assumed to be constructed with spherical grains characterized by a mean diameter, d_g. As fast as the bed clogs, the parameters B, ϵ, and k vary with the retention σ.

The nature of the fluid motion will be characterized by the carrier fluid Reynolds number Re; for flow-through granular porous media the Reynolds number is often written

$$\mathrm{Re} = \frac{u_m d_g \rho_l}{6(1 - \epsilon_o)\eta_l}$$

The physicochemical properties are characterized by Φ. In the general case, it can therefore be written

$$k = \text{function } (\eta_l, \rho_l, u_m, y, \rho_S, d, d_g, B_o, \epsilon_o, \Phi, \sigma, \mathrm{Re})$$

This general function simplifies to different forms depending on the deep filtration types which are considered later.

General kinetic equations of deep filtration. *Effect of the suspension concentration.* It is usually assumed that the retention probabilities, k and k', or other related parameters (see Equation 24′), such as the filtration coefficient λ, are independent of the concentration y of the particles in suspension. This hypothesis, confirmed by experiments, assumes that the particles do not interact. Thus, the process is a kinetic one analogous to a first-order reaction in chemical kinetics. This hypothesis seems the more reasonable as the filtered suspensions are always dilute (y is typically of the order of 0.1%).

Relative significance of gravity and inertial forces. In a bed of grains of diameter d_g, the radius of trajectories is of the order of $1/2\ d_g$ and the centrifugal inertial force which moves the particles away from the streamlines is of the order

$$\pi \frac{d^3}{3} (\rho_S - \rho_l) \frac{u^2}{d_g}$$

where u is the mean interstitial fluid velocity and is equal to u_m/ϵ.

But particles are also subjected to the gravity force, $\pi d^3(\rho_S - \rho_l)g/6$, where g is the acceleration of gravity; therefore, the ratio of the inertial force to the gravity force is

$$\frac{2}{g d_g} \left[\frac{u_m}{\epsilon}\right]^2$$

In Table I, this ratio is calculated for several values of d_g and u_m with $\epsilon = 0.40$.

Generally, deep filtration is practiced with d_g near 500 or 1000 μ and with a laminar flow (u_m of the order of 0.1 cm/sec), therefore, according to Table I, inertial forces are perfectly negligible with regard to gravity forces, as is often assumed (*14, 20, 22, 28, 53, 60, 69*).

Relative significance of sedimentation with respect to bulk flow. The particles subjected to gravity are moving with respect to the carrier fluid with the sedimentation velocity v_S, this velocity can be calculated by setting the gravity force equal to the fluid drag (Stokes Law).

$$v_S = \frac{d^2}{18\eta_l} (\rho_S - \rho_l) g \qquad (13)$$

To compare the sedimentation velocity with the mean interstitial velocity of the fluid, the minimum flow rate u_m such that $v_S/u \leqslant 0.01$ is calculated for water ($\eta_l = 0.01$ P) for several values of d, and $(\rho_S - \rho_l)$, again assuming $\epsilon = 0.40$ (Table II).

The values in Table II show that for typical deep filtration conditions, sedimentation is negligible only for small particles (approx. 1 μ) regardless of their density. For larger particles ($d < 25\ \mu$), sedimentation is negligible only if $(\rho_S - \rho_l)$ is small and for particles larger

Table II. Minimum Flow Rate u_m (in Cm/Sec) Such That $u/v_S \geq 100$ ($\epsilon = 0.40$, $\eta_l = 0.01$ P)

$\rho_S - \rho_l$: d (G/cm^3)	1 μ	10 μ	25 μ	50 μ	75 μ	100 μ
0.5	0.0011	0.11	0.68	2.72	6.15	10.9
1	0.0022	0.22	1.36	5.44	12.3	21.8
2	0.0044	0.44	2.72	10.9	24.5	43.6
4	0.0087	0.87	5.44	21.8	49.0	87.2
6	0.013	1.31	8.16	32.6	73.6	130.8

Table III. Ratio (in Per Cent) of Diffusional Mean Path of Particles to Mean Path Due to Bulk Flow

($d_g = 1000$ μ, $\epsilon = 0.40$)

u_m:d (Cm/s)	0.01 μ	0.10 μ	1.0 μ
0.001	6.0%	1.9%	0.60%
0.010	1.9%	0.60%	0.19%
0.100	0.60%	0.19%	0.06%
1.00	0.19%	0.06%	0.02%

than 25 μ, sedimentation is always appreciable. Hall (*20*) has suggested that the effect of sedimentation upon λ_0 can be written by

$$\lambda_0 \propto \frac{v_S}{d_g u_m}$$

Relative significance of Brownian motion with respect to bulk flow. In Brownian motion, characterized by a diffusivity D_B, the quadratic mean $\bar{x}^2$ of the particle path in a time interval Δt is given by the classical law

$$\bar{x}^2 = 2D_B\Delta t \tag{14}$$

If it is assumed that this time interval is equal to the residence time of a particle in an elementary layer of bed depth equal to the grains diameter, the particle entrained by the bulk flow will have moved forward of a length $d_g = u\Delta t$.

The ratio, $\bar{x}/d_g$, which characterizes the diffusion significance with respect to bulk flow is

$$\frac{\bar{x}}{d_g} = \frac{\sqrt{2D_B\Delta t}}{u\Delta t} = \sqrt{\frac{2D_B}{ud_g}} \tag{15}$$

Note that D_B/ud_g is the inverse of a Peclet number.

D_B can be estimated from the Stokes-Einstein relation (*17, 28*).

$$D_B = \frac{\textcircled{H}h_B}{3\pi\eta_l d} \tag{16}$$

where

h_B = Boltzmann constant
$\textcircled{H}$ = absolute temperature

By substituting Equation 16 into Equation 15 we get

$$\frac{\bar{x}}{d_g} = \sqrt{\frac{2\textcircled{H}\epsilon h_B}{3\pi\eta_l d u_m d_g}} \tag{17}$$

In Table III, some values of this ratio are given for water at 20°C for $d_g = 1000$ μ and $\epsilon = 0.40$.

This table shows clearly that Brownian motion always is insignificant with respect to bulk flow and is negligible in normal deep filtration conditions. By comparison, in aerosol filtration the efficiency is proportional to $(D_B/u_m d_g)^{2/3}$ when diffusion is the main mechanism (*17, 73*); with deep filtration notations, this result would be written

$$\lambda_0 \propto \frac{1}{d_g^{5/3}(u_m\eta_l d)^{2/3}}$$

In the three preceding paragraphs, it has been assumed that the fluid velocity was the same everywhere in the porous medium and equal to the mean value $u = u_m/\epsilon$. However, in such a complex medium as a granular bed, there are areas where the fluid velocity is larger than the mean velocity and others where the velocity is smaller. Besides in the close proximity of grains, the fluid velocities diminish rapidly due to the laminar flow velocities distribution. Therefore, it is probable that the preceding conclusions are not valid everywhere in the porous medium; but as more accurate information is not available, the comparisons may only be made with mean values.

Significance of direct interception. Even if particles have the same density as fluid, they meet the filter medium when the streamlines they follow become nearer than $d/2$ from the grains surface; this process occurs in constrictions and in flow-past obstacles (Figure 3). The particles brought into contact with the filter medium will stop if there are some available retention sites.

Stein (*70*) has shown that the fraction of suspended particles which contact the wall is proportional to $(d/r_c)^2$ (here r_c is the constriction radius), if a parabolic velocity distribution is assumed. As r_c depends on d_g, it may be shown that $\lambda_0 \propto (1/d_g)(d/d_g)^2$. Stein has demonstrated the same result for a flow around a sphere of diameter d_g.

In aerosol filtration, the direct interception of particles by a cylinder fiber of diameter d_g may be expressed by an analogous relation (*17, 73*)

$$\lambda_0 \propto \frac{1}{d_g}\left(\frac{d}{d_g}\right)^2$$

Stein's assumption (parabolic velocity distribution) is probably not exact since bed pores are short with respect to their diameter. If a flat velocity profile is assumed, this relation becomes

$$\lambda_0 \propto \frac{1}{d_g}\left(\frac{d}{d_g}\right)$$

STRAINING IN CONSTRICTION. Consider a triangular constriction formed by three tangent spherical grains (Figure 4), a diameter limit exists above which no particle can penetrate the constriction. This limit is such that $d/d_g = 0.154$. It is also possible to imagine straining by several particles, for three particles the limit is $d/d_g = 0.10$ and for four particles $d/d_g = 0.082$.

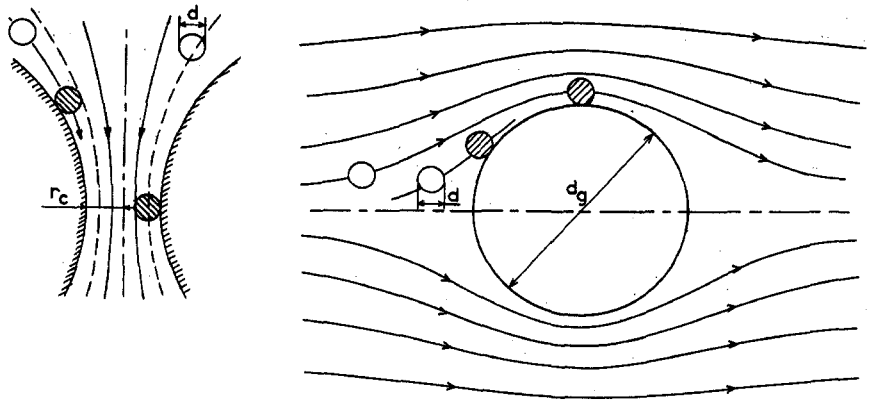


Figure 3. Direct interception

Straining in such a constriction is not necessarily the result of the simultaneous arrival of particles but may occur by successive stoppages of particles in each crevice. As the constriction is blocked, all particles entrained by the fluid are retained and fill up the upper cavern.

For the value of $d/d_g < 0.082$, it seems unlikely that constrictions are blocked, but particles may always become wedged in crevices.

WEDGING IN CREVICES. Consider the void between two tangent grains (Figure 5). Theoretically, a particle of any diameter may wedge in a crevice site; this interpretation is no longer valid if d/d_g is small because it can be assumed that the particle lies on a surface site.

For a particle to remain wedged in such a site, the fluid must bring it to the site with a sufficient kinetic energy.

To estimate the significance of this effect, the retention has been calculated in case all available sites are occupied on the first row.

Thus,

$$\sigma = \frac{1 - \epsilon_0}{2} \pi C \left(\frac{d}{d_g}\right)^2 \sqrt{\left(1 + \frac{d}{d_g}\right)^2 - 1} \qquad (18)$$

C is the coordination.

For $\epsilon_0 = 0.40$ and $C = 7.0$, the σ values are given in Table IV. These values show that the retention in crevice sites is important if $d/d_g \geq 0.05$.

To express the filtration coefficient λ_0 when particles are wedged in crevices, Hall (*20*) proposed the relation

$$\lambda_0 \propto \frac{1}{d_g} \left[\frac{d}{d_g}\right]^{3/2}$$

which is similar to relations discussed above for direct interception.

Surface forces significance. VAN DER WAALS FORCES. In the case of a particle and a porous bed grain, the interaction is modeled by a sphere and a flat plate, thus

Table IV. Retention Values When All Crevice Sites Are Occupied
($\epsilon_0 = 0.40$, $C = 7.0$)

d/d_g	0.02	0.05	0.08	0.10
σ	0.053%	0.53%	1.72%	3.02%

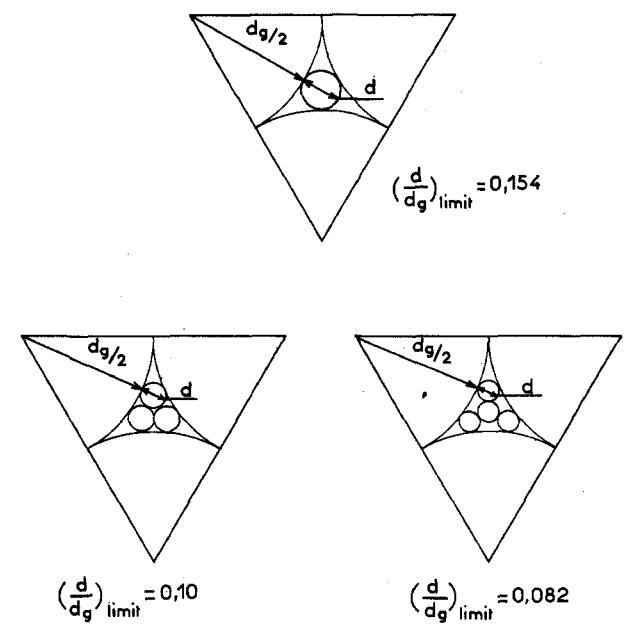


Figure 4. Straining in a triangular constriction

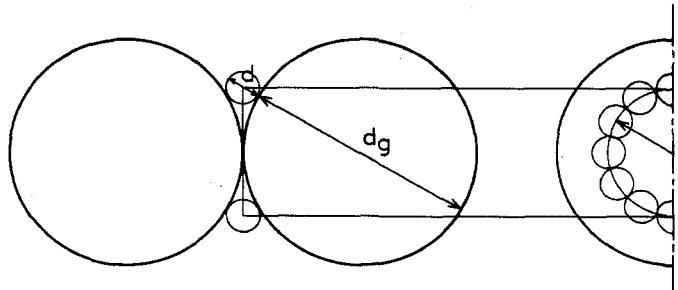


Figure 5. Retention in a crevice site

the attractive Van der Waals forces can be calculated by Equation 19 (*32*, *40*)

$$F_W = \frac{h_H d}{12r^2} \tag{19}$$

where h_H is the Hamaker constant and r is the separation between the particle and the grain.

Unfortunately, h_H determination is generally unreliable and only an order of magnitude is known (*40*). The range of these forces is smaller than 0.05 μ, typically the order of 0.01 μ; moreover Equation 19 is not correct if r is larger than 0.05 μ (*32*, *40*, *49*). These attractive forces decrease as $1/r^2$ and thus rather rapidly. To compare Van der Waals forces to gravity force, the separation has been calculated so that Van der Waals forces are equal to gravity force (Table Va). For particles suspended in water, $h_H = 10^{-13}$ erg is a reasonable order of magnitude; this separation is given by

$$r^2 = \frac{h_H}{2\pi g d^2(\rho_S - \rho_l)} \tag{20}$$

In spite of the fact that r is larger than the validity limit of the calculation, the results in Table Va indicate that for small particles ($d < 1$ μ), molecular forces dominate gravity force when the particles are close to the grains. By contrast, for larger particles ($d > 10$ μ) Van der Waals forces can be neglected, since gravity force varies as d^3 and molecular forces as d.

It is also interesting to compare Van der Waals forces with thermal energy forces. The suspended particles will be subjected to molecular forces if the Van der Waals forces energy is larger than the thermal energy of Brownian motion. As in the above discussion of Brownian motion with respect to bulk flow, the analogy with the gas kinetic theory may be continued because on the one hand, only an order of magnitude is desired and on the other hand, Van der Waals forces themselves are incompletely known. Therefore, it may be assumed that the mean kinetic energy of particles is $3h_B\textcircled{H}/2$, whereas the attraction energy is $h_H d/12r$. Consequently, the ratio of molecular attraction energy to kinetic energy of Brownian motion is $h_H d/(18\, r h_B \textcircled{H})$.

Several values of this ratio are given in Table Vb for $r = 0.01$ μ which is the order of magnitude of Van der Waals forces range. These values show that for particles of diameter larger than 0.1 μ the attraction energy overcomes the random-movement energy, but for smaller particles it is the contrary.

Although these forces act over a small range (of the order of 10^{-2} μ) some investigators (*22*, *49*, *50*) have built filtration models by assuming that particles are attracted towards grains by Van der Waals forces and move towards them at the limit velocity given by Stokes law. A suspension which is constantly agitated owing to the flow sinuosities would always bring particles into the range of these forces.

for Heertjes-Lerk (*22*)

$$\lambda_o \propto \frac{(1 - \epsilon_o)\epsilon_o h_H}{\eta_l u_m d_g^3}$$

for Mackrle (*49*, *50*)

$$\lambda_o \propto \frac{a_c}{\epsilon_o \, \mathrm{Re}^{\xi}}$$

where Re = Reynolds number
a_c = specific surface of the bed
ξ = a constant

According to experimental values

$$\lambda_o \propto \frac{\eta_l^{1/2}}{d_g^2 u_m}$$

Electrokinetic Forces—Ionic Double Layer. Any solid immersed in an electrolyte solution adsorbs some ions and by compensation, a second layer, richer in ions of opposite charge is formed around the solid. This second layer, known as the "diffuse layer," is bound to the solid and moves with it (*32*, *40*, *66*, *68*). The potential at the boundary between solution and second layer is the electrokinetic potential ζ and characterizes the electrokinetic interactions of the solid. The double layer thickness and the potential ζ depend on the ions present, their concentration, and on their valency. Thus, electrokinetic effects between particles and filter bed on the one hand, and between the particles themselves on the other hand, depend on pH, ionic strength of the solution, and the nature of the particles and ions (*32*, *40*).

The electrokinetic interaction force between a sphere and a flat plate is proportional to the sphere diameter and decreases exponentially with their separation (*40*). The range of electrokinetic forces is generally greater than that of Van der Waals forces but is not considered to exceed 0.1 μ (*32*, *40*). Electrokinetic forces are most often repulsive because most solids immersed in electrolyte solutions exhibit negative potentials ζ; but in some cases (*i.e.*, pH, ionic strength), for some solids, they may become attractive. Their order of magnitude is near that of molecular forces and depending upon the solids, they will be smaller or larger than Van der Waals

Table Va. Grain-Particle Separation (in μ) when Van der Waals Forces Are Equal to Gravity Force ($h_H = 10^{-13}$ erg)

$\rho_S - \rho_l$: d G/Cm^3	0.10 μ	1.0 μ	10 μ	100 μ
0.5	5.7	0.57	0.06	0.006
1.0	4.0	0.40	0.04	0.004
5.0	1.8	0.18	0.02	0.002

Table Vb. Ratio of Molecular Attraction Energy to Kinetic Energy of Brownian Motion ($h_H = 10^{-13}$ erg, $r = 0.01$ μ, H = 293°K)

d	0.01 μ	0.1 μ	1.0 μ	10 μ
Ratio	0.14	1.4	14	140

Table VI. Deep Filtration Types

Filtration type	*Particle size*	*Retention sites*	*Retention forces*	*Capture mechanisms*	*Spontaneous decolmatage*	*Provoked decolmatage*
Mechanical	$\geqslant 30\ \mu$	Constrictions, crevices, caverns	Friction, fluid pressure	Sedimentation, direct interception	Improbable	Flow direction reversal
Physicochemical	$\sim 1\ \mu$ ($<3\ \mu$)	Surface sites	Van der Waals forces, electrokinetic forces	Direct interception	Possible	Increase in flowrate
Colloidal	$<0.1\ \mu$	Surface sites	Van der Waals forces, electrokinetic forces, chemical bonding	Direct interception, diffusion	Possible	Increase in flowrate

forces. For each case, it is necessary to combine both surface effects to calculate the total effect (*32*, *40*). Physicochemical factors are important when small particles are filtered, because they can greatly influence the particle-bed interactions and the particle interactions (*via* flocculation).

The surface forces, molecular or electric, have small ranges compared to grain diameter; thus, they do not attract particles from the bulk of suspension towards the grains wall, but act only when particles are in close proximity of the surface.

Classification of Deep Filtration Types

Consideration of the mechanisms, which have been described in the preceding paragraphs, allows a classification according to filtered particle size (Table VI).

Large particles ($d \geqslant 30\mu$)—volume phenomena prevail over surface phenomena

Small particles (diameter of the order of 1 μ)—surface phenomena prevail

Colloidal particles ($d < 0.1\ \mu$)—the laws of the physical chemistry of colloids are most important

For particles with $3\ \mu < d < 30\ \mu$, volume phenomena and surface phenomena often have the same order of magnitude.

Analysis of Previous Work

Classification. The most important previous work is grouped according to the preceding classification by referring to the mean diameter of the particles used (Table VII).

Investigations with colloidal particles not first subjected to flocculation are not considered here. For each study it is also indicated whether flocculant or coagulant have been added because these additions may conceal some processes.

Flocculation. As previously discussed electrokinetic effects exist simultaneously between the filter bed and the particles themselves. A change in these effects may thus influence both particle-bed and particle-particle interactions. For instance, Smith (*67*, *68*) finds that only particle-bed interactions take place when particles are filtered without any coagulant. On the contrary, he notices the interactions between particles (flocculation) play a great part when coagulants or flocculants are added. In this case the particle-bed interactions are found to have no effect upon bed efficiency and the particle diameter is observed to increase. Ghosh (*18*, *19*), Borchardt-O'Melia (*4*) believe that adding certain ions causes flocculation and formation of larger particles which can be subjected to sedimentation or to straining between grains, whereas single particles are not. Stanley (*69*) and Stein (*70*), however, claim that flocculation does not occur inside the bed owing to the small particle size, low residence time, and small suspension concentrations. However, Camp (*6*), analyzing the results of Eliassen (*13*), considers that flocculation is probably an important mechanism. Finally O'Melia-Crapps (*60*) found their results to be better explained by bed-particle interactions than by particle interactions between themselves.

Elementary mechanisms. *Large particles.* Retention Sites. According to Delachambre (*10*) and Sakthivadivel (*63*), particles follow the streamlines but are stopped in the passageways too narrow for passage (crevices and constrictions). The resulting deposits continuously reduce the free passage and eventually cause blocking and filling up of the caverns above these passageways. Maroudas (*53*, *55*) assumes that particles settle by gravity on the grain tops or on the horizontal pore surfaces and thus narrow, or block the channels. They found the important parameter to be the ratio d/d_g; for $d/d_g > 0.15$, the porous medium is irreversibly blocked and a filter cake is formed; for $d/d_g < 0.065$, the retention remains always low and for the intermediate values a partial blocking of the porous bed may occur (this depends on particle shape and bed porosity). The bed becomes, in all cases, saturated—*i.e.*, it retains no more particles.

Evolution with Clogging. Sakthivadivel (*63*) shows that the retained matter improves the bed efficiency which is assumed to increase linearly with σ; but this effect is small and decreases further as d/d_g decreases from 0.11 to 0.07. Maroudas (*52*, *54*, *56*) believes there is a limit u_M for the interstitial velocity beyond which no particle may be retained because the deposit reduces the porosity, the interstitial velocity thus increases with σ, and, as a result, the efficiency decreases. Her proposed relation, $\lambda \propto 1/u - 1/u_M$, may also be written under a linear form $\lambda = \lambda_o (1 - b\sigma)$ where b is a constant.

Delachambre (*10*) assumes also that the clogging rate decreases linearly with the retention since it is assumed to

Table VII. Experimental Conditions of Some Investigations

Author	Particles: Type	Particles: Size (μ)	Porous Medium: Type	Porous Medium: Size (μ)	Liquid	Filtration rate (cm/s)	Flow	Flocculant or ions
			Large Particles					
Sakthivadivel	Styron	900–1400	Spherical plastic balls	12,500	Mineral oil	0.1–0.4	Laminar	None
Maroudas	Spherical polystyrene	125–390	Model filter	Channels > 3000	Water	0.06–30	Laminar and transitional	None
	Angular polystyrene	65–1100	Sintered disk	Channels > 200				
	Angular quartz	<50	Packed bed (glass spheres)	2000				
Delachambre	Spherical polystyrene	60–350	Glass rods Glass spheres	3300 2280–3600	Organic mixture		Turbulent and transitional	None
Leclerc	Pollen	32	Glass spheres	500–1000	Organic mixture	0.05–2.0	Laminar	None
Herzig	Pollen	31	Glass spheres	500	Organic mixture	0.02–0.1	Laminar	None
			Small Particles					
Edwards-Monke	Clay	1	Sand	350	Water	0.007	Laminar	Ions
Heertjes-Lerk	Iron hydroxide	0.1–10	Glass spheres	540, 650, 780	Water	0.14–0.3	Laminar	Flocculant
Ives	Polyvinylchloride	1.3	Glass spheres, anthracite sand	425–1100	Water	0.06–0.200	Laminar	Ions
Jorden	Clay	1	Gravel	5500	Water	0.007	Laminar	Ions
O'Melia Stumm	Ferric precipitate		Glass spheres	4000	Water	0.18	Laminar	Ions
Trzaska	Clay	2	Glass spheres	800, 1300	Water	0.007–0.025	Laminar	
			Mean Particles					
Borchardt, O'Melia	Algae	15–60	Sand	320, 400, 525	Water	0.01–0.14	Laminar	Without or with flocculant
Cleasby-Baumann	Hydrous ferric oxide, calcium carbonate	5 1–10	Sand	500	Water	0.07–0.7	Laminar	Flocculant
Eliassen	Ferric hydroxide	6–20	Sand	460	Water	0.14	Laminar	Flocculant
Fox-Cleasby	Ferric hydroxide	4–25	Sand	700	Water	0.2–0.4	Laminar	Flocculant
Ghosh	Diatomite	20	Sand, glass spheres	460, 650, 770	Water	0.07–0.2	Laminar	None
Herzig	Alumina, silica	4	Glass spheres	500	Water	0.05–0.3	Laminar	None
Hsiung	Ferric hydroxide		Sand	386, 458, 545, 649, 771	Water	0.4–0.8	Laminar	Flocculant
Ives	Algae	4 × 10, 5	Sand, glass spheres, anthracite	250–1300	Water	0.05–0.25	Laminar	None
	Kaolinite	2.5–10						
	Sepiolite	1–30						
	Diatomite	2–60						
	Quartz	2–22						
Iwasaki	Algae, clay, bacteria	1–40	Sand	100–800	Water	0.0035–0.012	Laminar	None
Krone	Bacteria	1 × 4	Sand	63, 190, 400, 1100	Water	0.01–0.20	Laminar	None
Ling	Diatomite	10	Sand	350–550	Water	0.07–0.40	Laminar	Flocculant
Mackrle	Ferric hydroxide, aluminum hydroxide	5–20	Sand, anthracite, calcium carbonate, glass spheres	900–1750	Water	0.10–0.400	Laminar	Flocculant
Mintz	Clay, humus, ferric hydroxide		Sand	1000–2150	Water	0.15–0.25	Laminar	Without or with flocculant
O'Melia-Crapps	Ferric hydroxide	20	Sand	700	Water	0.14	Laminar	Flocculant, ions
Ornatskii	Clay		Sand					
Stanford-Gates	Bacteria, alum floc	>10	Sand	500	Water	0.14	Laminar	Flocculant, surfactant
Shekhtman	Crayon paste	<10	Sand	800–1200				
Smith	Clay	5	Sand	600	Water	0.14	Laminar	Flocculant, ions
Stanley	Ferric hydroxide		Sand	250–750	Water	0.07–0.4	Laminar	Flocculant
Stein	Ferric hydroxide	10–20	Lucite rods Sand	600 450	Water	0.07–1.0 0.13–1.0	Laminar	Flocculant

be proportional to the unclogged volume fraction. Leclerc's (*41*) and Herzig's (*23*) work have also shown that the retention probabilities decrease as soon as clogging starts. Thus, it appears that the presence of retained particles in the bed decreases its efficiency.

Small particles (about 1μ). RETENTION SITES. Most investigators (*12, 22, 32, 36*) cite the importance of the specific surface of porous bed and agree that retained particles occupy surface sites.

RETENTION FORCES. Typically, only surface forces are considered. Heertjes and Lerk (*22*) consider Van der Waals forces but neglect electrokinetic forces, whereas in other conditions Jorden (*36*) assumes the contrary; for Ives (*29, 32*), Edwards and Monke (*12*), the attractive Van der Waals forces and the electrokinetic forces are taken into account.

CAPTURE MECHANISMS. Generally, sedimentation is neglected. Diffusion effects are also neglected for the particles of diameter $> 1\mu$. Ives (*34*), O'Melia, and Stumm (*61*) believe that particles are brought into contact with the filter medium by direct interception. Ives (*34*) takes λ_o to be proportional to specific surface, therefore to $1/d_g$, and inversely proportional to u_m and η_l^2. Edwards and Monke (*12*) and Heertjes and Lerk (*22*) assume that Van der Waals forces cause the contact between particles and filter bed, in spite of their small range.

EFFICIENCY EVOLUTION WITH RETENTION. Ives (*34*) found the bed efficiency increases with the retention when clogging begins, perhaps due to an increase in specific surface, then to decrease because the deposit finally deteriorates it. For others (*22, 71, 72*), this initial period is not observed. Ives (*34*) describes this evolution by

$$\lambda = \lambda_o\left(1 + b_1\sigma - \frac{b_1'\sigma^2}{\epsilon_o - \sigma}\right)$$

(b_1 and b_1' are constants). Heertjes and Lerk (*22*) believe that the decrease in λ is proportional to the increase in the interstitial velocity

$$\lambda = \lambda_o\left(1 - \frac{\beta\sigma}{\epsilon_o}\right)$$

while Trzaska (*71, 72*) assumes a linear decrease in k with respect to the retention.

Mean particles (3-30 μ). RETENTION SITES. Generally retained particles are considered to occupy only surface sites but some investigators suggest that the particles can be wedged between the grains in crevices or constrictions (*2, 6, 15, 20, 39*). Ghosh (*19*) and Ling (*43*) assume that retention occurs on grain surfaces or in crevices and sheltered areas where interstitial velocity is low.

RETENTION FORCES. When deposition on grain surfaces is assumed, only surface forces are usually considered. Mintz (*58*) and Mackrle (*49*) account for only Van der Waals forces, while others consider that these forces must be combined with electrokinetic forces (*29, 60, 69*). Sanford and Gates (*64*) assume that surface effects may be neglected when $d > 10\ \mu$.

CAPTURE MECHANISMS. Sedimentation is neglected by some (*16, 49, 60, 67, 68, 70*) but is considered a probable process by others (*6, 14, 15, 20, 28, 31, 39, 64*). Krone (*39*) assumes that sedimentation should not act upon the single particles but instead should act upon the agglomerates torn from the deposit; Diaper (*11*) and Ison (*27*) found it to be important.

Direct interception is considered by some authors (*14, 20, 27, 28, 49, 70*). Mackrle (*49, 50*) assumes that Van der Waals forces, in spite of their low range, are sufficient to explain the contacting of particles with bed grains. In addition to the expressions already mentioned for λ_o as a function of the experimental parameters (see the sections "General kinetic equations of deep filtration"), other relations are found in previous work; Mintz (*58*) finds the clogging rate to be proportional to $u_m^{0.3}/d_g^{1.7}$ at the initial instant. Ives (*57*) takes λ_o to depend on specific surface, interstitial velocity, fluid viscosity, and pores tortuosity

$$\lambda_o \propto \frac{1}{\eta_l u_m d_g^{1-3}}$$

The d_g exponent varies with experimental conditions.

Ison and Ives (*27*) find for a fixed gravitational parameter, v_s/u_m, and a given Reynolds number the bed efficiency decreases when the particle diameter increases according to $\lambda_o \propto (d/d_g)^{-2.3}$; this phenomenon is assumed to be due to hydrodynamic effects. Experimentally they observed

$$\lambda_o \propto \frac{1}{d_g}\left(\frac{d}{d_g}\right)^{-2.3}\left(\frac{v_s}{u_m}\right)^{1.3}\mathrm{Re}^{-2.7}$$

or

$$\lambda_o \propto \frac{\eta_l^{1.4} d^{0.3}}{d_g^{1.4} u_m^4}$$

According to Ison and Ives (*27*), hydrodynamic effects play a great part in the capture of particles.

Yao (*74*) studies theoretically the transport of suspended particles to porous medium grains. According to him interception and sedimentation play the most significant part in the capture of relatively large particles ($>5\ \mu$); for these particles the transport efficiency is proportional to d^2. On the contrary, for the submicroscopic particles ($<1\ \mu$), only diffusion intervenes and the transport efficiency is proportional to $d^{-2/3}$. For intermediate particles (1 or 2 μ) the transport efficiency reaches a minimum. Yao (*74*) really observes by experiment the existence of such a minimum. Iwasaki (*35*) found $1/(d_g u_m^{0.3})$ but Ives (*34*), taking again Iwasaki's experimental results, finds $\lambda_0 \propto 1/(d_g u_m)$.

EFFICIENCY EVOLUTION WITH RETENTION. The existence of an initial period during which the filtrate concentration decreases is often observed. This period is sometimes assumed (*2, 19, 35, 39*) to result from a narrowing of pores which favors the straining or the wedging of other particles; while at other times (*28, 31, 43, 70*), it is attributed to an increase in specific surface.

Generally, this initial effect is written under the form of a linear expression (*28, 31, 35, 62, 70*)

$$\lambda = \lambda_o(1 + b'\sigma)$$

Iwasaki (*35*) does not give a limit to this period, but the others consider that the interstitial velocity increases with the deposit and hinders particle retention; thus, the filtrate concentration curve *vs.* time presents a minimum.

Stein (*70*) proposes an expression for the final clogging stage

$$\lambda = \lambda_c \left(\frac{1 - b_2\sigma}{1 - b_2\sigma_c}\right)^4$$

where λ_c is the maximum value of λ which occurs when $\sigma = \sigma_c$ and b_2 is a constant.

Ives (*31*) postulates that the interstitial velocity increases and the tortuosity as well as the specific surface decrease at the same time. With these assumptions, he derives

$$\lambda = \lambda_o \left(1 + b_1\sigma - \frac{b_1'\sigma^2}{\epsilon_o - \sigma}\right)$$

He has later (*28*) generalized the relation by writing

$$\lambda = \lambda_o \left(1 + \frac{b''\sigma}{\epsilon_o}\right)^{\xi_1} \left(1 - \frac{\beta\sigma}{\epsilon_o}\right)^{\xi_2} \left(1 - \frac{\sigma}{\sigma_M}\right)$$

where b'', ξ_1, and ξ_2 are constants; σ_M is the maximum value of σ.

Ornastkii (*62*) found that taking this initial period into account does not change the results significantly and Ives (*30*) has also shown that it can be often neglected. Ornatskii (*62*) attributes the variations in k with σ to an increase in interstitial velocity and these variations are fairly well described by

$$k = k_o \frac{\epsilon}{\epsilon_o} \qquad \text{or} \qquad k = k_o \left(\frac{\epsilon}{\epsilon_o}\right)^2$$

Ives (*30*) takes the relation between λ and σ to be

$$\lambda = \lambda_o(1 - b_3^2\sigma^2)$$

Mackrle (*49*) first believed that the initial increase in λ did not exist because when clogging started, the decrease in porosity and in specific surface caused the retention probabilities to diminish and he proposed

$$\lambda = \lambda_o \left(1 - \frac{\beta\sigma}{\epsilon_o}\right)^{1.2}$$

Recently (*51*) he takes into account this initial period and suggests the relation

$$\lambda = \lambda_o \left(1 + 7.5\,\frac{\beta\sigma}{\epsilon_o}\right) \left(1 - \frac{\beta\sigma}{\epsilon_o}\right)^2$$

also based on variations in porosity and specific surface.

Mintz (*57, 58, 59*) states that the retained matter has an unequally strong structure. When the retained matter accumulates the interstitial velocity increases as well as the hydrodynamic forces. These forces become large enough to cause the breakaway of the retained particles which are less strongly attached to the others. However, the capture of suspended particles continues, and colmatage and decolmatage are simultaneous. According to Mintz the decrease in bed efficiency is due to a certain spontaneous decolmatage; the bed efficiency is maximum when colmatage starts and afterward decreases continuously.

Empirical Relations. Historically, experimental results have been grouped under empirical formulas. Unfortunately, these formulas often have been established with poorly defined parameters. Stanley (*69*) looked for a relation between d_g, u_m and the penetration—*i.e.*, the depth to which particles are retained—which was found by him to be proportional to $d_g^{2.46}u_m^{1.56}$.

Fair (*14*) noticed that the minimum depth required to produce a satisfactory filtrate in the usual conditions varies as

$$\frac{60}{\theta_F + 10}\, d_g^{5/3}$$

where θ_F is the Fahrenheit temperature.

Baylis (*2*) and Ling (*43*) proposed analogous formulas to calculate the time t_H at the end of which the pressure drop through the bed reaches a certain value. For Baylis

$$t_H \propto d_g^{2.15}u_m^{-1.50}$$

and for Ling

$$t_H \propto d_g^{2.32}u_m^{-1.48}$$

Hudson (*26*) considered that the length of filter run depends on $d_g^2\epsilon_o^3u_m^{-1.5}$.

Hsiung and Cleasby (*24*) recently proposed a new approach to the problem when analysing the experimental results of Eliassen (*13*) and Ling (*43*). They noticed the curves $(1/y_i)(\partial y/\partial z)$ against z have the same shape as the χ^2 distribution curves provided that the time (in hours) and the degree of freedom in χ^2 distribution are identified. Using this analogy they succeeded in grouping their experimental results in two curves. They found for a given suspension the grain diameter and the flow rate are connected by the relation $u_m d_g^{2.1} = c^t$ if the effluent quality and the bed depth are fixed. They also noticed the difference between the fluid pressure drop at the beginning and at the end of the operation is proportional to $u_m^{1.5}y_i^{1.4}z^{0.5}d_g^{-1.9}t$; this result is quite similar to those above mentioned.

Decolmatage. *Large particles.* It has been observed that the spontaneous decolmatage of large particles (diameter $\geqslant 30\ \mu$) is improbable in usual deep filtration conditions (*23, 52, 63*). Even when the fluid flow rate is increased by as much as a factor 3 or 4, decolmatage is low and sometimes nonexistent (*23, 63*). "Impulses"—*i.e.*, sudden variations of pressure or flow rate—are sometimes not observed to give any effect. When the same impulse is repeated several times and decolmatage occurs, it occurs only with the first impulse (*23*). On the contrary, flow direction reversal causes an appreciable decolmatage (*23, 41, 63*). Delachambre (*10*) has shown that it was possible to reach an equilibrium between colmatage and decolmatage in transitional and turbulent

flow, but his work is hardly comparable to deep filtration processes.

Mean and small particles. With particles of diameter < 30 μ, a consensus does not exist in the literature, with some experiments showing spontaneous decolmatage and others showing its impossibility. Mintz (*57*, *59*) reports the following experiments: if deep filtration is performed with a constant flow rate (u_m about 0.23 cm/sec) and with sand grains (d_g = 1500 μ), fine flakes are observed to emerge from the bed. After 2 hr, deep filtration is stopped. At this instant, the pressure drop of the fluid through the bed is found to be almost 25 times larger than at the filtration beginning. Then clear liquid is passed through the clogged bed at the same rate as the suspension; no flake is carried out by the fluid. However, if the flow rate is increased (about 30%) a certain decolmatage of previously retained particles is observed and many flakes emerge from the bed. If the flow rate is increased six or seven times further by progressive stages, a decolmatage occurs at each stage. Mintz deduces from these experiments the retained matter has an unequally strong structure and any increase in interstitial velocity and therefore, in pressure drop causes a certain decolmatage due to an increase in hydrodynamic forces. In particular, the capture of suspended particles causes a decrease in porosity and an increase in pressure drop. Consequently, colmatage and decolmatage are simultaneous. In Mintz's opinion, spontaneous decolmatage occurs during deep filtration. However, Stanley (*69*) found that radioactive particles of the same kind (hydroxide floc) used by Mintz, which were retained when filtration began, remained at their place during the whole operation. Mackrle (*49*) noticed that clear liquid flow at the same flow rate did not cause decolmatage in a clogged bed. With a microscope, Stein (*70*) observed the decolmatage of the least-linked flocs when interstitial velocity increased. Others (*2*, *15*, *39*, *43*) believe that decolmatage can occur when the interstitial velocity exceeds a certain limit. Ives (*57*) claims that it is unreasonable that particles, retained in the bed in spite of the hydrodynamic forces, are torn off by the same forces. Heertjes and Lerk (*57*) are also of this opinion. The authors (*23*) have also studied the behavior of small and mean particles retained in the bed. They are not broken away as the bed is fed with clear liquid at the same flowrate as the suspension, but in case of impulses, a great part of them is torn off.

These experimental results show the flow of clear liquid through a clogged bed at the same rate as the suspension does not cause any decolmatage. However, notable increases in interstitial velocity are observed to cause a certain decolmatage. Spontaneous decolmatage will occur during a deep filtration operation if such variations actually take place in the bed. It seems improbable that such increases really occur when colmatage starts because the flow is hardly disturbed by the retained matter. Thus, spontaneous decolmatage is normally improbable at the beginning of filtration, but when the bed is nearly saturated and when the interstitial velocity is considerably increased, decolmatage may occur. Typically, deep filtration is halted far before the bed is saturated (otherwise, the loss of pressure head would be too high and the filtrate much too turbid). Thus, it seems reasonable to assume that spontaneous decolmatage does not occur in normal deep filtration.

Importance of decolmatage study. It is obvious that knowledge of decolmatage conditions of a clogged bed is important because if they are accurately determined, they show how the particles are retained in bed. The behavioral difference between the large and small particles in the case of an impulse has just been seen; it shows that small particles are less strongly fixed in bed than large ones. Thus, the colmatage mechanisms are different in these two cases: large particles appear to be irreversibly fixed in the porous medium, whereas small particles seem to be fixed in a provisional way.

Decolmatage probability. Generally it is assumed for spontaneous decolmatage that k_r' is independent of σ and y and that it remains constant during the whole experiment (*3*, *44*, *58*, *65*). For example, Mintz (*58*) observes that k_r' is proportional to u_m/d_g but Delachambre (*10*) finds that the decolmatage rate is proportional to $(\sigma - \sigma_n)$ and not to σ. $(\sigma - \sigma_n)$ represents the particles which can be broken away and σ_n designates the particles that are irreversibly fixed.

Comparison of some experimental values of λ_o. When values of λ_o, calculated from various filtration experiments with particles of a diameter between 1 and 1000 μ with grains of a diameter between 200 and 12500 μ, are compared (Table VIII), it is seen that they are spaced between 2 and 0.002 cm^{-1} and principally between 0.5 and 0.005 cm^{-1}—that it is to say in an interval of two orders of magnitude. This shows that the porous medium efficiency does not vary much with the experimental parameters in spite of the diversity of the possible mechanisms.

However, it is necessary to note that this efficiency is sensitive to the nature of the bodies present, especially when the surface effects play a great part. For example, under the same conditions of grain diameter, of particles, and of flowrate, Ives (*32*) found that λ_o varies in the ratio of 1 : 3 according to the dissolved ions.

Kinetic Equation

If it is assumed that spontaneous decolmatage ("declogging") is improbable and that the retention probabilities, k and k', are independent of the suspension concentration (as assumed by most authors and verified by experiments), Equations 8 and 9 can be written

$$\frac{\partial \sigma}{\partial t} = k u_m y \qquad (8')$$

and

$$\frac{\partial \sigma}{\partial t} = k' \epsilon y \qquad (9')$$

where k and k' depend on the parameters which characterize the system (see above) and particularly on the retention σ. Differential Equations 8′ and 9′ are analogous with classical equations of chemical kinetics and

Table VIII. Some Experimental Values of λ_o

Author	Particles: Type	Particles: Size (μ)	Porous Medium: Type	Porous Medium: Size (μ)	Filtration rate (cm/sec)	λ_o (cm^{-1})
			Large Particles			
Sakthivadivel	Styron	1060	Spherical, plastic balls	12500	0.0985	0.015
		900		12500	0.0985	0.010
Maroudas	Angular polystyrene	65	Glass spheres	2000	0.63	0.026
		65		2000	2.50	0.007
		65		2000	8.85	0.002
		90		2000	2.50	0.012
		90		2000	10.0	0.004
		125		2000	2.28	0.021
		125		2000	8.35	0.013
			Small Particles			
Ives	Polyvinylchloride (PVC)	1.3	Glass spheres	460	0.060	0.024
				460	0.060	0.033
				460	0.060	0.070
				460	0.140	0.020
				460	0.200	0.140
	(PVC)	1.3	Anthracite	460	0.060	0.067
				460	0.140	0.30
				460	0.200	0.20
	(PVC)	1.3	Sand	425	0.060	0.110
				600	0.140	0.044
Iwasaki	Colon bacillus	1	Sand	227	0.0035	0.100
				227	0.0058	0.095
				227	0.0081	0.080
				104	0.0035	0.512
				104	0.0058	0.480
				104	0.0081	0.450
			Mean Particles			
Eliassen	Ferric hydroxide	6–20	Sand	460	0.140	0.200
Fox-Cleasby	Ferric hydroxide	4–25	Sand	700	0.270	0.10
				700	0.140	0.0060
Ives	Algae	5	Sand	540	0.140	0.14
				700	0.140	0.147
	Quartz	2–22	Anthracite	770	0.140	0.064
				1020	0.140	0.074
Iwasaki	Algae	15	Sand	829	0.0035	0.55
				829	0.0081	0.52
				227	0.0035	1.99
				227	0.0081	1.40
Mackrle	Ferric hydroxide	10	Calcium carbonate	1330	0.140	0.10
				1330	0.280	0.044
Maroudas	Angular quartz	<20	Glass spheres	2000	0.25	0.017
				2000	0.36	0.010
				2000	0.80	0.004

for this reason are called the kinetic equations of clogging. In fact, they form one kinetic equation only:

$$\frac{\partial \sigma}{\partial t} = Ky \tag{21}$$

Here K is called the clogging rate constant; the dimension of K is T^{-1} and K is related to k and k' by Equation 22

$$K = ku_m = k'\epsilon \tag{22}$$

Thus, K depends on the same parameters as k and k', especially on the retention. To take into account this fact, K can be written

$$K = K_o F(\sigma) \tag{23}$$

Such a kinetic equation has been proposed previously by some authors (*38*, *45*, *48*, *62*, *72*) and has also been derived from statistical considerations (*24*, *46*).

When writing Equation 21 with the retention age τ, Equation 21′ of unchanged form is obtained

$$\frac{\partial \sigma}{\partial \tau} = Ky \tag{21'}$$

Combining Equation 21′ with mass balance Equation 5′ yields

$$\left(\frac{\partial y}{\partial z}\right)_\tau = \frac{-K}{u_m} y \quad \text{or} \quad \left(\frac{\partial y}{\partial z}\right)_\tau = -ky \tag{24}$$

Kinetic equations analogous to Equation 24 are often proposed (*20*, *22*, *27*, *28*, *35*, *49*, *54*, *70*) but written with the variables, z and t, in the form:

$$\left(\frac{\partial y}{\partial z}\right)_t = -\lambda y \tag{24'}$$

where λ is the filtration coefficient introduced for the first time by Iwasaki (*35*), λ has the dimension L^{-1} and it is assumed to be independent of y but a function of retention σ and other parameters which characterize the system.

In all cases, Equation 24′ cannot be a kinetic equation because by eliminating $(\partial y/\partial z)_t$, between it and exact Equation 4 of mass balance, one obtains

$$\frac{\partial(\sigma + \epsilon y)}{\partial t} = \lambda u_m y$$

which is a kinetic equation between the whole quantity of particles (moving and retained) and the suspension concentration, whereas only retained particles must be taken into account for clogging.

However, if the first approximation is justified (*i.e.*, if ϵy is negligible with respect to σ, or if t and τ are very close), Equation 24′ becomes the kinetic equation of clogging since it can be written:

$$\frac{\partial \sigma}{\partial t} = \lambda u_m y$$

which is identical with Equation 21 and shows that λ becomes identical with k, the retention probability per unit of bed depth.

Recently, another type of kinetic equation has been proposed (*63*)

$$\left(\frac{\partial y}{\partial z}\right)_t = -\lambda_o[1 + b_4(\sigma + \epsilon y)]y \qquad (25)$$

where b_4 is a constant.

This equation contains the same error as the equations of type 24′. Moreover, λ depends on the total volume of particles (moving and retained) whereas in reality only retained particles can change the retention conditions.

However, if ϵy is small with respect to σ, the kinetic equations of the types 24′ and 25 are identical with type 21; all these equations have nearly the same form and are grounded on the hypothesis that spontaneous decolmatage (declogging) is improbable and retention probabilities are independent of the concentration.

If colmatage (clogging) and decolmatage (declogging) are simultaneous and if it is assumed as usual that k_c' is independent of y and k_r' is independent of σ and y, the kinetic equation may be written

$$\frac{\partial \sigma}{\partial t} = k_c'\epsilon y - k_r'\sigma \qquad (26)$$

Here $k_c'\epsilon y$ is the clogging rate, $k_r'\sigma$ the decolmatage rate.

Such a kinetic equation has previously been proposed (*3, 10, 44, 47, 65*). In addition to the assumptions discussed so far regarding $k_c'\epsilon$ and k_r' as functions of the experimental parameters, it is interesting to note that some authors (*3, 10, 44, 47*) have also assumed that k_c' is independent of σ although Shekhtman (*65*) assumes that $k_c'\epsilon$ does not depend on σ.

It is also noted that the kinetic equation proposed by Mintz (*57, 58*) under the form

$$\left(\frac{\partial y}{\partial z}\right)_t = -b_5 y + b_5'\sigma \qquad (27)$$

(where b_5 and b_5' are constants) contains the same error as the equations of type 24′.

In spite of their apparent simplicity, these kinetic equations of colmatage-decolmatage phenomena are very difficult to integrate and their application to real problems is not straightforward. Thus, as the categorical proof of the existence of spontaneous decolmatage has not yet been demonstrated (see "Decolmatage"), it does not appear advantageous to continue their study.

INTEGRATION OF THE DIFFERENTIAL SYSTEM

It has been seen that the most frequently used kinetic equation is of the first order and may be written

$$\frac{\partial \sigma}{\partial t} = Ky \qquad (21)$$

or

$$\frac{\partial \sigma}{\partial \tau} = Ky \qquad (21')$$

with

$$K = K_o F(\sigma) \qquad (23)$$

From this equation another differential equation is now derived.

Differential Equation in σ

Consider a porous bed, of depth Z, at the instant t_1. At $t = 0$, suspension begins to penetrate the bed and thence at $t = 0$, $\sigma = 0$ for all z.

The total mass balance for the time interval $[0, t_1]$ is given by Equation 28, (flow in) = (flow out) + (accumulation):

$$\int_o^{t_1} u_m y_i dt = \int_o^{t_1} u_m y(Z,t)dt + \int_o^Z [\sigma(z,t_1) + \epsilon(z,t_1)y(z,t_1)]dz \qquad (28)$$

At the depth Z, the kinetic equation can be written in the following form

$$y(Z,t) = \frac{1}{K_o F[\sigma(Z,t)]}\frac{\partial \sigma(Z,t)}{\partial t}$$

therefore

$$\int_o^{t_1} u_m y(Z,t)dt = \int_o^{t_1} \frac{u_m}{K_o F[\sigma(Z,t)]}\frac{\partial \sigma(Z,t)}{\partial t}dt = \int_o^{\sigma(Z,t_1)} \frac{u_m}{K_o}\frac{d\sigma(Z,t)}{F[\sigma(Z,t)]}$$

Then Equation 28 becomes

$$\int_o^{t_1} u_m y_i dt = \int_o^{\sigma(Z,t_1)} \frac{u_m}{K_o}\frac{d\sigma(Z,t)}{F[\sigma(Z,t)]} + \int_o^Z [\sigma(z,t_1) + \epsilon(z,t_1)y(z,t_1)]dz \qquad (29)$$

Equation 29 is established for a bed of fixed depth Z, but if it is derived with respect to Z (by replacing Z by z), it becomes

$$0 = \frac{u_m}{K_o}\frac{1}{F[\sigma(z,t_1)]}\frac{\partial \sigma(z,t_1)}{\partial z} + \sigma(z,t_1) + \epsilon(z,t_1)y(z,t_1)$$

since in steady operation u_m is the same in all bed sections and since in deep filtration u_m is kept constant during the experiment. This differential equation is valid for an arbitrary instant t_1, and therefore for every point (z,t)

$$\frac{\partial \sigma}{\partial z} = -\frac{K}{u_m}(\sigma + \epsilon y) = -\frac{K_o F(\sigma)}{u_m}(\sigma + \epsilon y) \qquad (30)$$

This differential equation in σ is very interesting and presents the following characteristics:

(1) It is valid whatever the function $F(\sigma)$ is, so long as the function $F(\sigma)$ does not become zero. This condition is always realized since $F(\sigma) = 0$ means that no separation occurs. Thus, Equation 30 is exact for any expression of K provided that K is independent of y, as assumed when a kinetic equation of type 21 is used.

(2) It is also valid if y_i changes with the time as long as K_o does not vary with y_i, which will be true if K is independent of y.

(3) It is exact with respect to the mass balance because no assumption has been made to derive it, except the hypothesis that diffusion is negligible. The Equation 30 becomes simpler in the first and second approximations.

(4) In the second approximation, previously discussed (see section on Approximate Equations), ϵ_o replaces ϵ, thus Equation 30 becomes

$$\frac{\partial \sigma}{\partial z} = -\frac{K}{u_m}(\sigma + \epsilon_o y) \qquad (31)$$

Furthermore with a change of variables (*i.e.*, if τ is substituted for t), Equation 31 becomes

$$\frac{\partial \sigma}{\partial z} = \frac{\epsilon_o}{u_m}\frac{\partial \sigma}{\partial \tau} - \frac{K}{u_m}(\sigma + \epsilon_o y)$$

and according to kinetic Equation 21′, it can be written

$$\left(\frac{\partial \sigma}{\partial z}\right)_\tau = -\frac{K}{u_m}\sigma \qquad (32)$$

(5) In the first approximation, all moving particles are neglected, (*i.e.*, ϵy is neglected with respect to σ), thence, Equation 30 becomes

$$\left(\frac{\partial \sigma}{\partial z}\right)_t = -\frac{K}{u_m}\sigma \qquad (32')$$

Differential Equations 32 and 32′ have exactly the same mathematical form but in Equation 32, the variables are z and τ and in 32′ z and t; one equation may be written for both cases

$$\frac{\partial \sigma}{\partial z} = -\frac{K}{u_m}\sigma \qquad (35)$$

without specifying t or τ.

Differential System

Thus, clogging is described by three equations

$$\frac{\partial \sigma}{\partial \tau} = -u_m\frac{\partial y}{\partial z} \quad \text{local mass transfer} \qquad (5')$$

$$\frac{\partial \sigma}{\partial \tau} = Ky = K_o F(\sigma)y \quad \text{kinetic equation} \qquad (21')$$

$$\frac{\partial \sigma}{\partial z} = -\frac{K}{u_m}\sigma \qquad (33)$$

The boundary conditions of the differential system are:

$$\sigma = 0 \quad \text{for} \quad (z,\tau) = (z,0)$$

$$y = y_i \quad \text{for} \quad (z,\tau) = (0,\tau)$$

The first condition means that there are no retained particles at z prior to the suspension arrival and the second that the suspension concentration in the inlet layer is that of the suspension applied to the filter.

Relation between σ, y, σ_i and y_i

Equations 24 and 33 have the same mathematical form:

$$\frac{1}{y}\frac{\partial y}{\partial z} = -\frac{K}{u_m} \qquad (24)$$

$$\frac{1}{\sigma}\frac{\partial \sigma}{\partial z} = -\frac{K}{u_m} \qquad (33)$$

Integrating, $\ln y = \ln \sigma + \text{constant}$.

The constant may be determined from the conditions in the bed layer at any z (*i.e.*, $z = 0$ is the inlet layer conditions σ_i, y_i). Thus, constant $= \ln (y_i/\sigma_i)$ and

$$\frac{y}{y_i} = \frac{\sigma}{\sigma_i} \qquad (34)$$

where σ_i is the retention in the inlet layer.

Equation 34 applies when the following conditions y and σ are taken at the same depth z, and y, y_i, σ, and σ_i are taken at the same retention age τ, therefore at different instants t such as

$$t - \frac{\epsilon_o z}{u_m} = \text{constant}$$

Equation 34 allows the determination of distribution of the concentrations y along the bed from that of the retentions σ, or the reverse, as long as y and σ are known in a reference layer (for example y_i and σ_i).

This equation has been previously derived (*22*, *30*) but only in particular cases. Here it is general and independent of the function $F(\sigma)$.

Existence of a Constant Clogging Front

If σ is plotted against z, for various retention ages, the curves are parallel (Figure 6). Equation 33 states that the retention curves, σ against z, have an identical slope at fixed σ. These curves are consequently parallel and can be deduced from each other by translations parallel to the axis $0z$. Thus, clogging of porous media progresses as a constant form front, analogous to a fully developed chromatography front.

Front velocity. The front velocity is determined from mass balance Equation 5′ which can be written under the form

$$\left(\frac{\partial z}{\partial \tau}\right)_\sigma = u_m\frac{\partial y}{\partial \sigma}$$

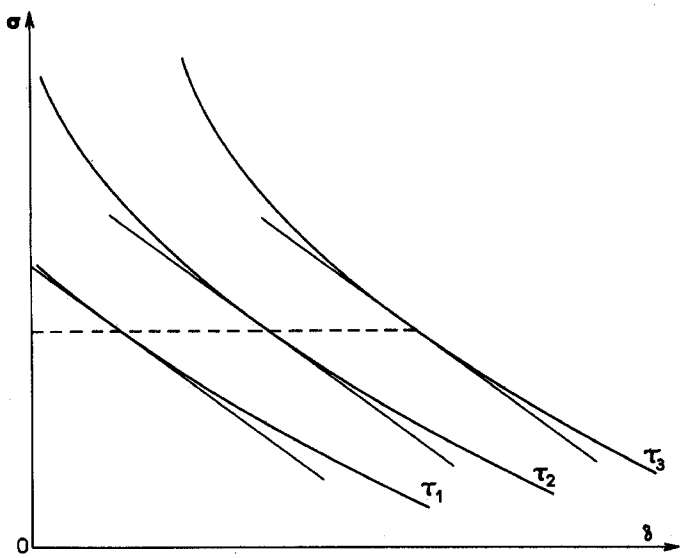


Figure 6. Retention curves

or

$$\left(\frac{\partial z}{\partial \tau}\right)_\sigma = u_m \frac{\partial y}{\partial z}\frac{\partial z}{\partial \sigma}$$

Using Equations 24 and 33 and noting that $(\partial z/\partial \tau)_\sigma$ is the front velocity V_F, we see that $V_F = u_m(y/\sigma)$ or from Equation 34

$$V_F = u_m \frac{y}{\sigma} = u_m \frac{y_i}{\sigma_i} \tag{35}$$

From Equations 21′ and 33, it can be seen that

$$V_F = -\frac{\partial \sigma/\partial \tau}{\partial \sigma/\partial z} \tag{36}$$

Equation 36 states that the partial derivatives of σ with respect to τ and z are proportional all along a curve of σ against z at a fixed retention age τ.

These properties are strictly applicable only to curves of σ against z at fixed τ. As the operating time increases, t and τ approach each other and the above properties can be applied to the retention curves at fixed t.

The existence of such a constant clogging front is only visible on the curves of σ against z. The plots of y against z do not present this feature since $\partial y/\partial z$ depends on both y and σ. However, if y_i and σ_i are constant (as happens when the inlet layer has reached saturation), the curve of y against z will have a constant form, since according to Equation 34, this curve can be deduced by proportionality to the retention curve of σ against z.

Differential System Integration

Integration with respect to z. Equation 33 $\partial\sigma/\partial z = -K\sigma/u_m$ is integrated to obtain

$$\frac{\partial \sigma}{\partial z} = -\frac{K_o}{u_m} F(\sigma)\sigma \tag{33}$$

or

$$\frac{d\sigma}{\sigma F(\sigma)} = -\frac{K_o}{u_m} dz$$

thence

$$\int \frac{d\sigma}{\sigma F(\sigma)} = -\frac{K_o}{u_m} z + c'$$

The dependence of the constant upon τ is determined by substituting an expression of σ as a function of τ for a fixed-bed layer. This reference may be the inlet layer as taken for Equation 34 or another layer such as the outlet layer. The inlet layer is usually more convenient since the concentration of the suspension applied to the filter is well known.

Thus, an implicit function between σ and σ_i is obtained:

$$\varphi(\sigma, \sigma_i) = -\frac{K_o}{u_m} z$$

If it can be resolved, σ can be written under the form

$$\sigma = F_1\left(\sigma_i, \frac{K_o}{u_m} z\right) \tag{37}$$

Determination of σ_i. σ_i is calculated from kinetic Equation 21′

$$\frac{d\sigma_i}{d\tau} = K_o F(\sigma_i) y_i$$

$$\frac{d\sigma_i}{F(\sigma_i)} = K_o y_i d\tau$$

from which

$$\sigma_i = F_2(K_o y_i \tau)$$

where σ_i depends only on the dimensionless term $K_o y_i \tau$.

Expression of σ. By substituting the expression of σ_i into Equation 37, σ is obtained:

$$\sigma = F_1'\left(K_o y_i \tau, \frac{K_o}{u_m} z\right) \tag{38}$$

here σ depends only on two dimensionless terms, $K_o y_i \tau$ and $(K_o/u_m)z$. If y_i changes with the time, Equation 38 is still valid if $y_i \tau$ is replaced by $\int_o^\tau y_i dt$

Determination of y. According to Equation 34, $y = y_i(\sigma/\sigma_i)$. Thus, y/y_i, like σ, is only dependent on both dimensionless terms $K_o y_i \tau$ and $(K_o/u_m)z$.

Numerical integration. The total analytic resolution of the differential system can only be made for some single functions $F(\sigma)$ (see Appendix 1). This function is always inferred from experiments since it is impossible to obtain it theoretically. The inferred function is often complex and total analytic resolution is difficult. Even if the first stage of the calculation is easy (integration of $1/F(\sigma)$) the second is less easy with the integration of $1/(\sigma F(\sigma))$ and the resolution of the implicit function $\varphi(\sigma, \sigma_i) = -z(K_o/u_m)$. However, from the expression of σ_i as a function of τ and from the unresolved implicit

function, the retention curve (σ against z) can be numerically determined. For any value of τ, σ_i can be calculated and for every value of σ_i, z can be determined with the implicit function for various $\sigma \leqslant \sigma_i$. In other words, the clogging front is determined at fixed τ.

Flow Rate Effect

It is assumed that the flow rate alters only K_o and that $F(\sigma)$ is independent of u_m, the clogging front curves for any flowrate $(u_m)_2$ can be deduced from that of a different flowrate $(u_m)_1$, since according to Equation 33 the ratio of their tangent slopes is constant at fixed σ and is equal to $(K_o/u_m)_1(u_m/K_o)_2$.

In the same way, curves of K against σ can also be deduced from each other.

Effect of Suspension Concentration

At fixed flow rate u_m. As σ and the ratio y/y_i depend only on two dimensionless terms, $K_o z/u_m$ and $K_o y_i t$ (or $K_o \int_o^\tau y_i dt$ if y_i varies with the time), all experimental points performed under identical conditions, except for various concentrations, y_i must lie on the same curves σ and y/y_i against $\int_o^\tau y_i dt$ at fixed z. The colmatage is thus only dependent on the quantity of particles which have entered the bed for a fixed flowrate.

At variable flow rate. If experiments made under the same conditions except for different flowrates and concentrations are considered, they can be conveniently represented on the same diagram by plotting the curves of $(K_o/u_m)z$ against $K_o \int_o^\tau y_i dt$ at fixed σ or at fixed y/y_i since σ and y/y_i depend only on these two terms. Should K_o be proportional to $u_m{}^{\xi'}$ (ξ' is a constant), it would be sufficient to plot $z u_m{}^{\xi'-1}$ against $u_m{}^{\xi'} \int_o^\tau y_i dt$ at fixed σ or y/y_i.

These properties are only exact if K_o is independent of y_i, as usually observed, and if $F(\sigma)$ is independent of u_m.

Experimental points performed with various values of other parameters, such as d_g, ϵ_o, η_l, will also lie on the same curves $\sigma = c^t$ and $y/y_i = c^t$ when plotting $(K_o/u_m)z$ against $K_o y_i t$ provided that these parameters alter only K_o and do not intervene in $F(\sigma)$.

Conclusion

These properties which have been derived above are valid whatever the form of the function $F(\sigma)$. Therefore, they apply to all deep filtration cases, provided that the kinetic equation is similar to Equation 21′—*i.e.*, of first order—and that spontaneous decolmatage does not occur. It has been assumed that particle diffusion as well as a part of the suspended particles (second approximation) are negligible, but these assumptions are generally verified by experiment and consequently, these results apply to all common deep filtration cases. They characterize the kinetic equation of first order without decolmatage and diffusion.

PRESSURE DROP

Experimental Results—Empirical Formulas

In constant rate filtration, the pressure drop P across the filter increases with the clogging. Various investigators (*9*, *22*, *25*, *28*) have proposed purely empirical formulas to represent the influence of the retention σ on the pressure drop through a filter layer Δz sufficiently thin such that the retention may be considered constant in it.

Most of these formulas may be written

$$\frac{\Delta P}{(\Delta P)_o} = \frac{1}{(1 - j\sigma)^m} \tag{39}$$

where ΔP is the pressure drop through the element of thickness Δz, and σ is the retention. $(\Delta P)_o$ is the pressure drop across a thickness Δz of clean porous bed, and j and m are constants.

Expanding Equation 39 when $j\sigma$ is small,

$$\frac{\Delta P}{(\Delta P)_o} = 1 + mj\sigma + \frac{m(m+1)}{2} j^2\sigma^2 + \ldots \tag{40}$$

Experience indicates that often only the first term of the expansion is significant and thus that ΔP increases linearly with σ.

Ives (*30*), Heertjes and Lerk (*22*) have shown under certain conditions that if

$$\frac{\Delta P}{(\Delta P)_o} = 1 + jm\sigma \tag{41}$$

the total pressure drop P across the entire filter is in certain conditions a linear function of time. At the beginning of filtration through a deep filter, the filtrate concentration, y_f, is very low. It may be assumed that all the particles are retained in the filter. If the initial velocity, u_m, and the initial concentration, y_i, remain constant in time, the total retention in the bed at time t is

$$u_m y_i t = \int_0^Z \sigma dz \tag{42}$$

Since $P = \int_0^Z (dP/dz)dz = P_o/Z \int_0^Z (1 + mj\sigma)dz$, the pressure drop may be written

$$P = P_o\left(1 + \frac{mju_m y_i t}{Z}\right) \tag{43}$$

This last expression is easier to verify experimentally than Equation 41, since the knowledge of σ is not required in Equation 43.

Cleasby and Baumann (*9*), Eliassen (*13*) and Ling (*43*) have observed a linear increase of the pressure drop through the whole bed with time. Experimentally Mintz (*58*) finds that the proportionality coefficient, here denoted by mj, varies as $d_g{}^{0.5}/u_m$, Heertjes and Lerk (*22*) observe it to be independent of d_g but proportional to $1/u_m$ whereas Hsiung and Cleasby (*24*) find it to be proportional to $(y_i{}^{0.4} Z^{0.5} d_g{}^{0.1})/u_m{}^{0.5}$. Stanley (*69*) observed that the pressure drop increased linearly with the quantity of particles which are retained but notes that the proportionality coefficient varied from one experi-

Table IX. Some Experimental Values of (mj)

Author	y_i	Particles, d	u_m, cm/mn	$P_o/\rho_l g z$	Grains d_g/mm	Grains Type	mj
Mackrle	$0.3\ 10^{-3}$	Iron hydroxide	8.2	0.14	1.33	Calcium carbonate	57.8
		10 μ	16.7	0.215			28.4
Ives	$0.135\ 10^{-3}$	Chlorella	8.2	0.65	0.54	Sand	61.5
	$0.055\ 10^{-3}$	5 μ	8.2	0.44	0.70		432
Robinson	$0.38\ 10^{-3}$	Quartz powder	8.2	0.57	0.77	Anthracite	41.2
	$0.76\ 10^{-3}$	2 to 22 μ	8.2	0.35	1.02		79.3

ment to another. He ascribes this variation to a variation of the entrance concentration of particles. Borchardt and O'Melia (*4*) obtained nearly linear variations when coagulants were not used in their suspensions. Heertjes and Lerk (*22*), Ives (*31*) find Equation 43 invalid except at the beginning of filtration, when all particles are retained. Ives (*30*) calculated values of $P_o jm/Z$ from his own work and compared his results with those of Mackrle (*49*) and Robinson. Table IX gives the corresponding values of (jm) calculated from Ives (*30*) and the experimental conditions.

We note that values of (jm) are widespread over a considerable range, possibly due to the diversity of experimental conditions. Because Equation 39 does not indicate any dependence on other conditions, a more realistic model is now considered—the Kozeny-Carman model of clogging beds.

Utilization of Kozeny-Carman Model

In the domain of laminar flow the pressure drop $(\Delta P)_o$ through a clean layer Δz of porous bed can be written with the fundamental expression of Kozeny-Carman (*8*):

$$\frac{(\Delta P)_o}{\Delta z} = (h_K)_o \eta \frac{u_m}{\epsilon_o} \frac{(a_c)_o^2}{\epsilon_o^2} \tag{44}$$

in which u_m is the suspension approach velocity, η the dynamic viscosity of the suspension, $(a_c)_o$ the bed specific surface (total grain surface per unit bed volume), ϵ_o porosity of clean porous bed, $(h_K)_o$ the Kozeny constant.

This relation is based on the use of simple model in which the void space is described by a bundle of identical, cylindrical, nonconnected capillaries. In this model,

$$(h_K)_o = 2\gamma T^2 \tag{45}$$

where γ is a circularity coefficient and T a tortuosity coefficient. If the bed specific surface is replaced by the grain specific surface $(a_g)_o$,

$$(a_g)_o = \frac{(a_c)_o}{1 - \epsilon_o}$$

the Kozeny-Carman relation becomes Equation 44′

$$\frac{(\Delta P)_o}{\Delta z} = (h_K)_o \eta u_m (a_g)_o^2 \frac{(1 - \epsilon_o)^2}{\epsilon_o^3} \tag{44'}$$

By changing the rate through a clogged bed, Cleasby and Baumann (*9*) verified that the pressure drop remains proportional to the fluid velocity. The pressure drop may be thus written

$$\frac{\Delta P}{\Delta z} = h_K \eta \frac{u_m}{\epsilon} \frac{(a_c)^2}{\epsilon^2} \tag{46}$$

in which ϵ is the new porosity of the clogged bed, a_c the actual or fictitious bed specific surface, and h_K the new value of Kozeny constant.

From Equations 44 and 46,

$$\frac{\Delta P}{(\Delta P)_o} = \frac{h_K}{(h_K)_o} \left[\frac{a_c}{(a_c)_o}\right]^2 \left(\frac{\epsilon_o}{\epsilon}\right)^3 \tag{47}$$

To obtain a theoretical relation between ΔP and σ, it is necessary to establish first the relations between the retention σ and the Kozeny constant, the bed specific surface a_c, and the porosity. The decrease in porosity can be easily calculated by:

$$\frac{\epsilon}{\epsilon_o} = 1 - \frac{\beta\sigma}{\epsilon_o} \tag{48}$$

where β is the inverse of compaction factor of retained particles. The two other factors of Equation 47 are difficult to determine. If it is assumed that the Kozeny constant does not vary with retention, then

$$\frac{\Delta P}{(\Delta P)_o} = \left[\frac{a_c}{(a_c)_o}\right]^2 \left(\frac{\epsilon_o}{\epsilon}\right)^3 \tag{49}$$

It remains to derive the variation of bed specific surface with the retention.

The first hypothesis assumes that the Kozeny model is not simply a mathematical or fictitious model but that porous media are actually constituted of bundles of capillaries. It is further supposed that the particles diameter, d, is small compared to the pore diameter, d_p, and that the capillaries are uniformly coated internally by the deposit.

If there are N capillaries of length Z_p and diameter d_p in the unit volume, the initial porosity is

$$\epsilon_o = N Z_p \frac{\pi d_p^2}{4} \tag{50}$$

If deposited particles diminish the diameter of the capillaries to $(d_p - \Delta d_p)$, the porosity becomes

$$\epsilon = N Z_p \frac{\pi (d_p - \Delta d_p)^2}{4} \tag{51}$$

or

$$\epsilon = \epsilon_o \left(1 - \frac{\Delta d_p}{d_p}\right)^2 \tag{52}$$

Similarly, the initial bed specific surface is

$$(a_c)_o = NZ_p\pi d_p$$

and with deposition it becomes

$$a_c = NZ_p\pi(d_p - \Delta d_p) \tag{53}$$

or

$$a_c = (a_c)_o\left(1 - \frac{\Delta d_p}{d_p}\right) \tag{54}$$

combining Equations 51 and 54,

$$\frac{a_c}{(a_c)_o} = \left(\frac{\epsilon}{\epsilon_o}\right)^{1/2} \tag{55}$$

or

$$\frac{a_c}{(a_c)_o} = \left(1 - \frac{\beta\sigma}{\epsilon_o}\right)^{1/2} \tag{56}$$

With the aid of Equations 48, 49 and 56, the pressure drop as a function of retention, bed porosity, and particle property, β is given by Equation 57

$$\frac{\Delta P}{(\Delta P)_o} = \left(1 - \frac{\beta\sigma}{\epsilon_o}\right)^{-2} \tag{57}$$

Empirical Equation 39 is thus verified with

$$m = 2$$

$$j = \frac{\beta}{\epsilon_o}$$

and thus,

$$mj = \frac{2\beta}{\epsilon_o} \tag{58}$$

Unfortunately, such a result is incompatible with values of (mj) presented in Table IX. β is the coefficient by which it is necessary to multiply the retention σ to obtain the fraction of porous medium which does not participate to the flow of suspension. In the case of a regular and uniform coating of pores, the maximum value of β corresponds more or less to the value for a cubic packing arrangement of spheres, which is

$$\beta = \frac{6}{\pi} = 1.91$$

This value is too small, since for a porosity of 0.40, it should lie in the range $6 < \beta < 87$. Is it possible to improve the Kozeny model?

Generalization from Uniform Clogging of Surface Sites to Case of Arbitrary Texture

In the Kozeny-Carman model, the bed specific surface a_c decreases when the retention σ increases since the pores are concave volumes. However, if void space was included between nonjoined spheres, the void volumes would be convex and the specific surface would increase with σ.

Table X presents some expressions for $a_c/(a_c)_o$ and $\Delta P/(\Delta P)_o$ as functions of σ for the cases of different geometrical porous media which are regularly and uniformly coated. We have included in the table the value of $a_c/(a_c)_o$ for $\beta\sigma/\epsilon_o = 0.1$. In the case of spherical joined grains, the hypothesis of uniform coated retention does yield a decrease of bed specific surface when σ increases and the same value of $a_c/(a_c)_o$ is found (0.96 instead of 0.95) as in the Kozeny-Carman model. This coincidence does not imply similarity of the models.

Generalizing, the hypothesis of retention by uniform coating may be written:

$$\frac{a_c}{(a_c)_o} = \left[1 + \frac{\beta\sigma}{f(\epsilon_o)}\right]^{m_1} \tag{59}$$

where m_1 may vary between 1/2 and 2/3 and $f(\epsilon_o)$ between $-\epsilon_o$ and $1 - \epsilon_o$. Substituting Equation 59 in the Kozeny-Carman relation,

$$\frac{\Delta P}{(\Delta P)_o} = \left(1 - \frac{\beta\sigma}{\epsilon_o}\right)^{-3}\left[1 + \frac{\beta\sigma}{f(\epsilon_o)}\right]^{2m_1} \tag{60}$$

Equation 60 does not account for variation of tortuosity. The term which describes the variation of bed specific surface varies less rapidly than the term describing the variation of porosity. Investigators agree neither on which model should be chosen nor on the importance of the variation of specific surface. Some take the model of cylindrical pores of Kozeny, others use the model of nonjoined spheres, and still others assume that specific surface does not vary.

Kozeny's model. In his thesis, Mackrle (*49*) used a model of narrowed pores but proposed replacing the exponent 1/2 of Equation 56 by a constant m_2 such that

$$\frac{a_c}{(a_c)_o} = \left(1 - \frac{\beta\sigma}{\epsilon_o}\right)^{m_2} \tag{61}$$

and therefore,

$$\frac{\Delta P}{(\Delta P)_o} = \left(1 - \frac{\beta\sigma}{\epsilon_o}\right)^{2m_2 - 3} \tag{62}$$

Experimentally, Mackrle found $m_2 \simeq 1$.

More recently, Mackrle (*51*) proposed another relation between bed specific surface and retention:

$$\frac{a_c}{(a_c)_o} = \left(1 - \frac{\beta\sigma}{\epsilon_o}\right)^{1.5}\left(1 + 7.5\,\frac{\beta\sigma}{\epsilon_o}\right)^{0.5} \tag{63}$$

In this expression, the first factor describes the reduction of bed specific surface when pore diameter decreases and the second the increase of specific surface when the spherical grains are being coated. Values of the coefficients are determined by experiment.

Ives and Pienvichitr (*33*) utilized Mackrle's Equation 61 but their experimental results yield $m_2 \simeq -4$. It is noted that m_2 has a value far different from that found by Mackrle and the sign of m_2 means that the specific surface increases. This is at variance with the hypothesis of the pore model in which specific surface is assumed to decrease when retention increases.

Nonjoined spheres model. Stein (*70*) and Camp (*7*) proposed a model for spherical grains which are uniformly coated, obtaining:

$$\frac{a_g}{(a_g)_o} = \frac{1}{1 + \dfrac{\beta\sigma}{3(1 - \epsilon_o)}} \tag{64}$$

The denominator in Equation 64 is the first term in the expansion of

$$\left(1 + \frac{\beta\sigma}{1 - \epsilon_o}\right)^{1/3}$$

and therefore,

$$\frac{a_g}{(a_g)_o} = \left(1 + \frac{\beta\sigma}{1 - \epsilon_o}\right)^{-1/3} \tag{65}$$

where a_g is the grain specific surface, by definition

$$a_g = a_c \cdot \frac{1}{1 - \epsilon} \tag{66}$$

Combining Equations 65, 66, and 48,

$$a_c = (a_c)_o \left(1 + \frac{\beta\sigma}{1 - \epsilon_o}\right)^{2/3} \tag{67}$$

which is also given in Table X. Stein (*70*) showed that his model is in agreement with Eliassen's results (*13*). He also tried to take into account the proximity of the grains which forbids a uniform coating. His calculation yields equations which do not represent the experimental results well and he concludes that the equations of Kozeny are not particularly well suited to describe a clogged bed.

Maroudas (*52*, *56*) worked with particles of relatively large dimensions (>25 μ) and also used the model of nonjoined spheres. She described the pressure drop by

$$\frac{\Delta P}{(\Delta P)_o} = \left(\frac{1 - \epsilon}{1 - \epsilon_o}\right)^{4/3} \left(\frac{\epsilon_o}{\epsilon}\right)^3 \tag{68}$$

or

$$\frac{\Delta P}{(\Delta P)_o} = \left(1 + \frac{\beta\sigma}{1 - \epsilon_o}\right)^{4/3} \left(1 - \frac{\beta\sigma}{\epsilon_o}\right)^{-3} \tag{69}$$

Other models. Sakthivadivel (*63*) studied the retention of large particles, and proposed two models in which the tortuosity remains constant. One is a model of uniform coating of parallel pores, but the expression that he gives:

$$\frac{\Delta P}{(\Delta P)_o} = \left(\frac{\epsilon_o}{\epsilon}\right)^3 \tag{70}$$

does not correspond to his hypothesis, since it does not take into account the variation of bed specific surface. Equation 70 is experimentally verified for very fine particles (*63*).

The other is an "hydraulic radius model." In this model, it is assumed that the grain diameter, d_g, remains constant, and thus that the grain specific surface is constant. Then

$$\frac{\Delta P}{(\Delta P)_o} = \left(\frac{1 - \epsilon}{1 - \epsilon_o}\right)^2 \left(\frac{\epsilon_o}{\epsilon}\right)^3 \tag{71}$$

or

$$\frac{\Delta P}{(\Delta P)_o} = \left(1 + \frac{\beta\sigma}{1 - \epsilon_o}\right)^2 \left(1 - \frac{\beta\sigma}{\epsilon_o}\right)^{-3} \tag{72}$$

This model gives a better representation of Sakthivadivel's experimental results than Equation 70.

Ghosh (*18*, *19*) and Ling (*43*) have also used the "hydraulic model" since they assume that grain specific surface varies very slowly whereas $(1 - \epsilon)^2/\epsilon^3$ varies rapidly.

Mintz (*57*) assumed that the bed specific surface, a_c, remains constant and used Equation 70.

In conclusion, even when investigators have agreed to adapt the fundamental Kozeny-Carman relation to

Table X. Expressions of Bed Specific Surface and Pressure Drop as Function of Retention σ

Models	$\frac{a_c}{(a_c)_o}$ *vs.* σ	$\frac{a_c}{(a_c)_o}$ for $\frac{\beta\sigma}{\epsilon_o} = 0.1$, $\epsilon_o = 0.4$	$\frac{\Delta P}{(\Delta P)_o} = f(\beta,\sigma,\epsilon_o)$	*Limit expansion of* $\frac{\Delta P}{\Delta P_o}$
Circular capillaries	$\frac{a_c}{(a_c)_o} = \left(1 - \frac{\beta\sigma}{\epsilon_o}\right)^{1/2}$	$\frac{a_c}{(a_c)_o} = 0.95$	$\frac{\Delta P}{(\Delta P)_o} = \left(1 - \frac{\beta\sigma}{\epsilon_o}\right)^{-2}$	$\frac{\Delta P}{(\Delta P)_o} = 1 + \frac{2\beta\sigma}{\epsilon_o} + \ldots$
Spherical caverns	$\frac{a_c}{(a_c)_o} = \left(1 - \frac{\beta\sigma}{\epsilon_o}\right)^{2/3}$	$\frac{a_c}{(a_c)_o} = 0.94$	$\frac{\Delta P}{(\Delta P)_o} = \left(1 - \frac{\beta\sigma}{\epsilon_o}\right)^{-5/3}$	$\frac{\Delta P}{(\Delta P)_o} = 1 + 1.67\,\frac{\beta\sigma}{\epsilon_o} + \ldots$
Hyperbolic paraboloid	$\frac{a_c}{(a_c)_o} = 1$	$\frac{a_c}{(a_c)_o} = 1$	$\frac{\Delta P}{(\Delta P)_o} = \left(1 - \frac{\beta\sigma}{\epsilon_o}\right)^{-3}$	$\frac{\Delta P}{(\Delta P)_o} = 1 + 3\,\frac{\beta\sigma}{\epsilon_o} + \ldots$
Void between cylindrical nonjoined rods	$\frac{a_c}{(a_c)_o} = \left(1 + \frac{\beta\sigma}{1 - \epsilon_o}\right)^{1/2}$	$\frac{a_c}{(a_c)_o} = 1.033$	$\frac{\Delta P}{(\Delta P)_o} = \left(1 - \frac{\beta\sigma}{\epsilon_o}\right)^{-3} \times \left(1 + \frac{\beta\sigma}{1 - \epsilon_o}\right)$	$\frac{\Delta P}{(\Delta P)_o} = 1 + 3.67\,\frac{\beta\sigma}{\epsilon_o} + \ldots$
Void between nonjoined spherical grains	$\frac{a_c}{(a_c)_o} = \left(1 + \frac{\beta\sigma}{1 - \epsilon_o}\right)^{2/3}$	$\frac{a_c}{(a_c)_o} = 1.044$	$\frac{\Delta P}{(\Delta P)_o} = \left(1 - \frac{\beta\sigma}{\epsilon_o}\right)^{-3} \times \left(1 + \frac{\beta\sigma}{1 - \epsilon_o}\right)^{4/3}$	$\frac{\Delta P}{(\Delta P)_o} = 1 + 3.89\,\frac{\beta\sigma}{\epsilon_o{}^a} + \ldots$
Void between spherical joined grains coordination C^b	$\frac{a_c}{(a_c)_o} = \left(1 + \frac{\beta\sigma}{1 - \epsilon_o}\right)^{2/3} \times \left(1 - \frac{C}{2}\,\frac{\beta\sigma}{3(1 - \epsilon_o) + \beta\sigma}\right)$	$\frac{a_c}{(a_c)_o} = 0.96$	See annexes	$\frac{\Delta P}{(\Delta P)_o} = 1 + 2.11\,\frac{\beta\sigma}{\epsilon_o{}^a} + \ldots$

[a] $\epsilon_o = 0.40$.
[b] $C = 8$.

the case of a clogged bed, they have differed in the choice of the model to be used. The various solutions which they propose, do not give any more information than the purely empirical expression Equation 39. Their models are all based on the assumed regular and uniform coating of the interior surfaces of the porous medium. It seems unreasonable that clogging occurs only on surfaces sites, especially for the case of large particles in the suspensions.

Nonuniform Clogging

In the section on Generalization from the Uniform Clogging of Surface Sites to the Case of Arbitrary Texture, we tried eliminating a restriction to the Kozeny-Carman model. We replaced the uniformly concave surface of the capillaries by various types of concave or convex surfaces.

There still remains another restriction. In reality, the porous medium is not composed of pores of uniform size. It would be better represented by a group of interconnected pores which present widenings or constrictions, or better by a combination of caverns connected ones to the others by longitudinal and transverse constrictions (Figure 7).

In the longitudinal constrictions, fluid velocity will be higher. The pressure drop will depend essentially on the number and shape of these constrictions. In the transverse constrictions and in the dead zones of the caverns, the fluid velocity is less. Then depending on whether the kinetic energy more or less favors capture processes, clogging will occur preferentially in the first or second zone. In these conditions, a model of uniform coating of the cavern surfaces is hardly credible and must be replaced by two different models.

Blocking of constrictions. In this case, the entire flowrate through a long canal can be stopped by the presence of one large grain in a constriction of this canal. Fluid then will be deflected and pass through parallel canals which will undergo an increased head loss.

This mode of clogging also may be represented as the clogging of a pile of perforated plates, supposing that the holes of the plates have nearly the same diameter as the suspension particles.

Heertjes (*21*), studying the blocking of cloth with particles verified the relation:

$$\frac{\Delta P}{(\Delta P)_o} = \frac{1}{\left(1 - m' \frac{\sigma}{N_p}\right)^{m''}} \tag{73}$$

where N_p is the number of pores per unit section and m' and m'' are constants.

Sakthivadivel (*63*) and Maroudas (*52*, *54*, *56*) also have a model of blocking pores to describe filtration of large particles ($d/d_q > 0.08$). Each time a pore is blocked in constant flow rate filtration, the interstitial fluid velocity, u, through the other pores increases. At every instant, interstitial velocity is inversely proportional to the number of pores which remain free.

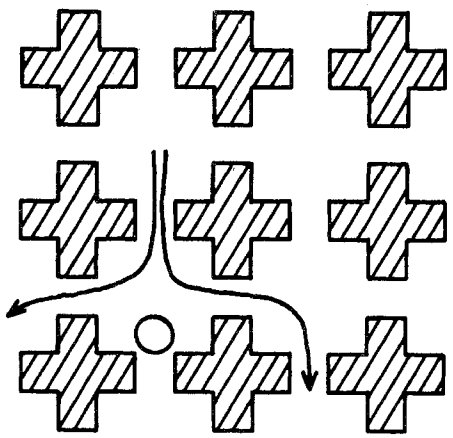


Figure 7. Model of caverns connected with constrictions

$$\frac{u}{u_o} = \frac{1}{1 - n_b} \tag{74}$$

where n_b is the fraction of pores blocked.

Assuming that the pressure drop is proportional to the interstitial fluid velocity (laminar flow),

$$\frac{\Delta P}{(\Delta P)_o} = \frac{1}{1 - n_b} \tag{75}$$

Stein (*70*) and Camp (*7*) have tried to apply this model of blocked pores to the retention of mean-sized (10 μ) particles. Stein was unsuccessful in applying it to Eliassen's (*13*) experiments, since he found that Equation 75 gave too small an increase of the pressure drop.

The problem lies in relating the fraction of blocked pores to the retention. The fraction of blocked pores corresponds to the decrease of the volume fraction which remains free to the fluid flow. By analogy with Equation 48, we write

$$\frac{\epsilon}{\epsilon_o} = 1 - n_b = 1 - \frac{\beta'\sigma}{\epsilon_o} \tag{76}$$

and

$$\frac{\Delta P}{(\Delta P)_o} = \frac{1}{1 - \frac{\beta'\sigma}{\epsilon_o}} \tag{77}$$

Here β' is the coefficient by which it is necessary to multiply σ to obtain the fraction of porous volume which no longer participates in the fluid flow. In the case of blocked pores which are entirely filled with particles, β' will be equal to β, the inverse of the compaction factor of the packing of particles in the pores. However, if only one particle is enough to block a canal, β' is then equal to the ratio of the canal volume to the particle volume. So β' will be difficult to determine since it will depend not only on the particle shape and size but also on the texture of the porous medium. Further, β' cannot be considered constant since the blocking of a pore by one or more particles does not exclude further deposition of particles in the dead zone created. These

particles, which do not affect the pressure drop, will modify the value of β'.

Hudson (*25*) did not take β' into account, and he proposed

$$\frac{\Delta P}{(\Delta P)_o} = \frac{1}{1 - \frac{\sigma}{\epsilon_o}}$$

Le Goff and Delachambre (*42*) consider that the pressure drop in a bed of spheres depends only on the passage of fluid through triangular, square, and pentagonal constrictions formed by the spheres. Large particles are retained in the caverns upstream of these constrictions which are progressively clogged beginning with triangular constrictions first. They calculate the number of triangular and square constrictions as well as the values of retention corresponding to the clogging of the caverns upstream of these constrictions. By summing the contribution to the velocity increase from each type of blocked constrictions they relate the pressure drop ΔP to the retention σ. Finally, their experimental results verify the derived expression for ΔP.

Deposition in dead zones. Dead zones do not participate in the flow of suspension. The presence of particles in these zones will not modify the pressure drop but will complicate the estimation of a relation between pressure drop and retention.

APPENDIX I

Analytic Integration of the Differential System for Some Simple Cases

$$\frac{\partial \sigma}{\partial \tau} = -u_m \frac{\partial y}{\partial z}$$

$$\frac{\partial \sigma}{\partial \tau} = K_o F(\sigma) y$$

$F(\sigma) = 1$, then $K = K_o$

$$\sigma = K_o y_i \tau \exp\left(-\frac{K_o}{u_m} z\right)$$

$$y = y_i \exp\left(-\frac{K_o}{u_m} z\right)$$

$F(\sigma) = 1 - b\sigma$ (b is a constant)

$$\sigma = \frac{1}{b} \frac{1 - \exp(bK_o y_i \tau)}{1 - \exp\left(\frac{K_o}{u_m} z\right) - \exp(bK_o y_i \tau)}$$

$$y = y_i \frac{\exp(bK_o y_i \tau)}{-1 + \exp\left(\frac{K_o}{u_m} z\right) + \exp(bK_o y_i \tau)}$$

$F(\sigma) = 1 - b^2\sigma^2$

$$\sigma = \frac{1}{b}\left[1 + (G^2 - 1) \exp\left(\frac{2K_o}{u_m} z\right)\right]^{-1/2}$$

$$y = y_i G\left[1 + (G^2 - 1) \exp\left(\frac{2K_o}{u_m} z\right)\right]^{-1/2}$$

with

$$G = \frac{1 + \exp(2bK_o y_i \tau)}{\exp(2bK_o y_i \tau) - 1}$$

Partial Analytic Integration for Some Other Cases

The analytic expression for σ_i and the implicit relation between σ and σ_i are given.

$F(\sigma) = (1 - b\sigma)^{1/2}$ (b is a constant)

$$\sigma_i = K_o y_i \tau \left(1 - \frac{1}{4} bK_o y_i \tau\right)$$

$$\ln \frac{\sigma}{\sigma_i} + 2 \ln \frac{1 + \sqrt{1 - b\sigma_i}}{1 + \sqrt{1 - b\sigma}} = -\frac{K_o}{u_m} z$$

$F(\sigma) = (1 - b\sigma)^{3/2}$

$$\sigma_i = \frac{K_o y_i \tau (4 + bK_o y_i \tau)}{(2 + bK_o y_i \tau)^2}$$

$$\ln \frac{\sigma}{\sigma_i} + 2 \ln \frac{1 + \sqrt{1 - b\sigma_i}}{1 + \sqrt{1 - b\sigma}} + 2\left[\frac{1}{\sqrt{1 - b\sigma}} - \frac{1}{\sqrt{1 - b\sigma_i}}\right] = -\frac{K_o}{u_m} z$$

$F(\sigma) = (1 - b\sigma)^2$

$$\sigma_i = \frac{K_o y_i \tau}{1 + bK_o y_i \tau}$$

$$\ln \frac{\sigma}{\sigma_i} + \ln \frac{1 - b\sigma_i}{1 - b\sigma} + \frac{1}{1 - b\sigma} - \frac{1}{1 - b\sigma_i} = -\frac{K_o}{u_m} z$$

$F(\sigma) = (1 - \sigma/\sigma_m)(1 - \sigma/\sigma_M)$ (σ_m and σ_M are constant)

$$\sigma_i = \frac{\sigma_m \sigma_M [1 - \exp(G')]}{\sigma_M - \sigma_m \exp(G')}$$

with

$$G' = \frac{\sigma_m - \sigma_M}{\sigma_m \sigma_M} K_o y_i \tau$$

$$\ln \frac{\sigma}{\sigma_i} + \frac{\sigma_m}{\sigma_M - \sigma_m} \ln \frac{\sigma_M - \sigma}{\sigma_M - \sigma_i} - \frac{\sigma_M}{\sigma_M - \sigma_m} \ln \frac{\sigma_m - \sigma}{\sigma_m - \sigma_i} = -\frac{K_o}{u_m} z$$

APPENDIX 2

Calculation of Variation of Bed Specific Surface with Retention for Different Models

Model of spherical caverns

We suppose that the porous medium is composed of N spherical caverns of diameter d per unit of bulk volume. By regular deposition on the walls of the caverns, the diameter diminishes to $(d - \Delta d)$ and the porosity changes

from $$\epsilon_o = N\frac{\pi d^3}{6} \tag{1}$$

to $$\epsilon = N\pi\frac{(d - \Delta d)^3}{6} \tag{2}$$

or $$\epsilon = \epsilon_o\left(1 - \frac{\Delta d}{d}\right)^3 \tag{3}$$

Similarly, the specific surface varies

from $$(a_c)_o = N\pi d^2 \tag{4}$$

to $$a_c = N\pi(d - \Delta d)^2 \tag{5}$$

or $$a_c = (a_c)_o\left(1 - \frac{\Delta d}{d}\right)^2 \tag{6}$$

Combining Equations 3 and 6 yields 7:

$$\frac{a_c}{(a_c)_o} = \left(\frac{\epsilon}{\epsilon_o}\right)^{2/3} \tag{7}$$

Model of nonjoined cylindrical rods

We suppose that the porous medium is composed by N cylindrical rods of diameter d and length Z per unit of bulk volume. By regular coating of the rods, the diameter increases to $d + \Delta d$, solid fraction of porous medium changes

from $$1 - \epsilon_o = NZ\frac{\pi d^2}{4} \tag{8}$$

to $$1 - \epsilon = NZ\pi\frac{(d + \Delta d)^2}{4} \tag{9}$$

or $$1 - \epsilon = (1 - \epsilon_o)\left(1 + \frac{\Delta d}{d}\right)^2 \tag{10}$$

In the same way, the specific surface goes

from $$(a_c)_o = NZ\pi d \tag{11}$$

to $$a_c = NZ\pi(d + \Delta d) \tag{12}$$

or $$a_c = (a_c)_o\left(1 + \frac{\Delta d}{d}\right) \tag{13}$$

Using Equations 10 and 13, we get

$$\frac{a_c}{(a_c)_o} = \left(\frac{1 - \epsilon}{1 - \epsilon_o}\right)^{1/2} \tag{14}$$

Model of nonjoined spherical grains

We suppose that the porous medium is composed of N_S spherical grains of diameter d_g per unit volume of filter. By regular coating of the grains, the solid fraction of the porous medium varies

from $$1 - \epsilon_o = N_S\frac{\pi d_g^3}{6} \tag{15}$$

to $$1 - \epsilon = N_S\pi\frac{(d_g + \Delta d_g)^3}{6} \tag{16}$$

or $$1 - \epsilon = (1 - \epsilon_o)\left(1 + \frac{\Delta d_g}{d_g}\right)^3 \tag{17}$$

and the specific surface varies

from $$(a_c)_o = N_S\pi d_g^2 \tag{18}$$

to $$(a_c) = N_S\pi(d_g + \Delta d_g)^2 \tag{19}$$

or $$a_o = (a_c)_o\left(1 + \frac{\Delta d_g}{d_g}\right)^2 \tag{20}$$

Combining Equations 17 and 20, we get

$$\frac{a_c}{(a_c)_o} = \left(\frac{1 - \epsilon}{1 - \epsilon_o}\right)^{2/3} \tag{21}$$

Model of spherical-joined grains (coordination C)

Figure 8 shows a cutaway view of two joined spherical grains which are partially coated. The coated thickness is $\Delta d_g/2$. The new grain volume is

$$\frac{\pi(d_g + \Delta d_g)^3}{6} - v \tag{22}$$

If there are C points of junction (coordination C), the new volume of a grain will be

$$\frac{\pi(dg + \Delta dg)^3}{6} - Cv \tag{23}$$

with $$v = \frac{\pi(\Delta dg)^2}{4}\left(\frac{dg + \Delta dg}{2} - \frac{1}{3}\frac{\Delta dg}{2}\right) \tag{24}$$

Thus the new solid fraction of the porous medium is

$$1 - \epsilon = \frac{N_S\pi(dg + \Delta dg)^3}{6} - \frac{N_S C\pi(\Delta dg)^2}{8}\left(dg + \frac{2}{3}\Delta dg\right) \quad (25)$$

$$\text{or} \quad 1 - \epsilon = \frac{N_S\pi dg^3}{6}\left[\left(1 + \frac{\Delta dg}{dg}\right)^3 - \frac{3C}{4}\left(\frac{\Delta dg}{dg}\right)^2 - \frac{C}{2}\left(\frac{\Delta dg}{dg}\right)^3\right] \quad (26)$$

Since Δdg is very small in comparison to dg, the corrective terms which take into account the proximity of grains are unnecessary and thus using Equation 15,

$$(1 - \epsilon) \simeq (1 - \epsilon_o)\left(1 + \frac{\Delta dg}{dg}\right)^3 \quad (27)$$

which is identical to Equation 17,

$$\text{or} \quad \frac{1 - \epsilon}{1 - \epsilon_o} \simeq 1 + \frac{3\Delta dg}{dg} \quad (28)$$

$$\text{or} \quad \frac{3\Delta dg}{dg} \simeq \frac{\beta\sigma}{1 - \epsilon_o} \quad (29)$$

Similarly, the new surface of a grain is

$$\pi(dg + \Delta dg)^2 - w \quad (30)$$

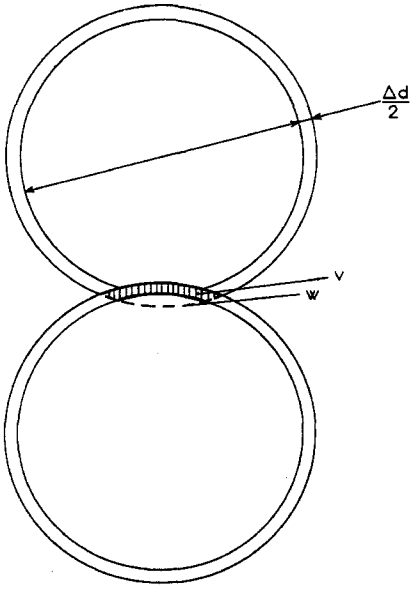


Figure 8. Cutaway view of two joined and coated grains

$$\text{with} \quad w = \frac{\pi(dg + \Delta dg)\,\Delta dg}{2} \quad (31)$$

For N_S grains with coordination C per unit volume of filter, the specific surface will be

$$a_c = N_S\pi(dg + \Delta dg)^2 - N_S C w \quad (32)$$

$$\text{or} \quad a_c = N_S\pi(dg + \Delta dg)^2\left[1 - \frac{C\Delta dg}{2(dg + \Delta dg)}\right] \quad (33)$$

Using Equation (18), we see that

$$a_c = (a_c)_o\left(1 + \frac{\Delta dg}{dg}\right)^2\left[1 - \frac{C\Delta dg}{2(dg + \Delta dg)}\right] \quad (34)$$

combining Equation 34 with Equations 29 and 27 yields Equation 35

$$a_c \simeq (a_c)_o\left(1 + \frac{\beta\sigma}{1 - \epsilon_o}\right)^{2/3} \times \left[1 - \frac{C}{2}\frac{\beta\sigma}{3(1 - \epsilon_o) + \beta\sigma}\right] \quad (35)$$

Since

$$\frac{\epsilon}{\epsilon_o} = 1 - \frac{\beta\sigma}{\epsilon_o} \quad (36)$$

and admitting that

$$\frac{\Delta P}{\Delta P_o} = \left[\frac{a_c}{(a_c)_o}\right]^2\left(\frac{\epsilon_o}{\epsilon}\right)^3 \quad (37)$$

and combining Equations 35, 36, and 37 yield Equation 38

$$\frac{\Delta P}{\Delta P_o} \simeq \left(1 - \frac{\beta\sigma}{\epsilon_o}\right)^{-3}\left(1 + \frac{\beta\sigma}{1 - \epsilon_o}\right)^{4/3} \times \left[1 - \frac{C}{2}\frac{\beta\sigma}{3(1 - \epsilon_o) + \beta\sigma}\right]^2 \quad (38)$$

Assuming that $\beta\sigma/\epsilon_o$ is very small in comparison to unity, we find that

$$\frac{\Delta P}{\Delta P_o} \simeq 1 + \frac{\beta\sigma}{\epsilon_o}\left[3 + \frac{4\epsilon_o}{3(1 - \epsilon_o)} - \frac{C\epsilon_o}{3(1 - \epsilon_o)}\right] \quad (39)$$

$$\text{or} \quad \frac{\Delta P}{\Delta P_o} \simeq 1 + \frac{\beta\sigma}{\epsilon_o}\left[3 + \frac{\epsilon_o(4 - C)}{3\,(1 - \epsilon_o)}\right] \quad (40)$$

Conclusion—Theory of Pressure Drop

A universal theory which deals with the problem of variation of pressure drop with retention does not exist. Two models may be used. The first states that there is an uniform and regular coating of internal surfaces of the porous filter. It may be adapted to filtration of small particles, but it always yields semiempirical results. The second model corresponds to the clogging of filters by large particles. Since a very good knowledge of the texture of the filter is needed, this latter model is more difficult to use.

Unfortunately, these models allow a description of the change of pressure drop with retention, for the experiment from which they are derived, but they do not allow an accurate prediction of how ΔP varies with σ in a new situation.

SUMMARY

The various studies about flow of suspensions through porous media allow the following conclusions:

1. The possible elementary mechanisms of deep filtration are now known, and they could be brought in evidence. However, it is always difficult to evaluate accurately their significance in any system. Therefore, experimental studies are needed before any filtration.

2. Nevertheless, two theoretical filtration types may be defined:

(*a*) a mechanical deep filtration for large particles (over 30 μ); for them, volume phenomena prevail and spontaneous decolmatage is improbable.

(*b*) a physicochemical deep filtration for small particles (approx. 1 μ); for them, surface effects prevail and spontaneous decolmatage may occur in case of sudden variations of flowrate or pressure.

Typical deep filtrations are performed for mean particles of an intermediary size; thus, volume phenomena and surface effects have the same order of magnitude.

3. However, in most cases, kinetic equation of colmatage has been verified by experiment to be of first order—*i.e.* there is no interaction between suspended particles.

4. Mass balance equation may be simplified in usual deep filtration. The third approximation (no diffusion) and the second approximation ($\epsilon \simeq \epsilon_0$) are valid for all common deep filtrations, whereas the first approximation is justified only if the retentions are large enough.

5. General macroscopic properties are deduced from kinetic equation of first order and mass balance equation; they are independent of the studied system and describe the evolution of colmatage in deep porous beds. They allow an easy representation of experimental results and a rapid calculation of industrial deep filter beds.

6. The fluid pressure drop through the porous medium increases with retention; various attempts have been made to describe this evolution:

(*a*) Empirical formulas are practical but do not explain what really occurs in porous media. Besides, the coefficients vary greatly from one investigator to another.

(*b*) Kozeny model accurately describes the flow of liquid through clean porous bed; however, it may not be applied to clogged beds because too small an increase in pressure drop is obtained when calculating the pressure drop from a model based on the uniform coating of cylindrical pores.

(*c*) The model based on the uniform coating of bed grains with retained particles is sometimes in agreement with experimental results in the case of small particles. However, a consensus does not exist between the investigators to explain the variations of specific surface.

(*d*) For large particles, a model based on the blocking of constrictions is preferable; however, it is difficult to apply this model because the volume occupied by a retained particle is unknown.

7. The study of colmatage and the study of pressure drop are connected as they are two aspects of the same phenomenon—*i.e.*, the retention of suspended particles smaller than the pores of the porous medium.

NOMENCLATURE

a_c	= bed specific surface
a_g	= grain specific surface
$b, b', b_1 \ldots$ etc.	= constants
d	= particle mean diameter
d_g	= grain mean diameter
d_p	= pore diameter (Kozeny model)
g	= gravity acceleration
h_B	= Boltzmann constant
h_H	= Hamaker constant
h_K	= Kozeny constant
j	= constant
k	= retention probability par unit depth
k'	= retention probability per unit time
k_c'	= capture probability per unit time
k_r'	= decolmatage probability per unit time
$m, m', m'', m_1, m_2, \ldots$	= constants
n_b	= fraction of blocked pores
q	= retention probability in a porous medium element
r	= grain-particle separation
r_c	= constriction radius
t	= time (variable)
t_H	= time at the end of which the pressure drop reaches a certain value (Baylis, Ling)
t_1	= particular value of t
u	= mean interstitial velocity of suspension
u_m	= suspension approach velocity
$(u_m)_1, (u_m)_2$	= particular values of u_m
u_M	= critical interstitial velocity (Maroudas)
v_S	= sedimentation velocity
$\bar{x}$	= mean diffusional path
y	= volume fraction of particles in suspension
z	= depth (variable)
B	= permeability
C	= coordination
D	= diffusion coefficient
D_B	= diffusivity (Brownian motion)
$F(\sigma)$	= function of σ in $K = K_0F(\sigma)$
F_1, F_1', F_2	= functions
F_w	= Van der Waals forces
G, G'	= dimensionless parameters
K	= clogging rate constant
N	= number of pores per unit volume of filter
N_p	= number of pores per unit of filter section
N_S	= number of spheres per unit volume of filter

P = total pressure drop across filter depth
Re = Reynolds number
T = tortuosity coefficient
V_F = front velocity
Z = filter bed depth
Z_p = pore length

Greek Letters

β = inverse of compaction factor (1 − porosity) of retained matter
β' = particle property
γ = circularity coefficient
ϵ = bed porosity
ζ = electrokinetic potential
η = dynamic viscosity of suspension
η_l = dynamic viscosity of carrier fluid
θ_F = fahrenheit temperature
λ = filtration coefficient
λ_c = maximum value of λ (Stein)
ξ, ξ', ξ_1, ξ_2 = constants
ρ_l = specific mass of carrier fluid
ρ_S = specific mass of particles
σ = retention—*i.e.*, volume of deposited particles per unit filter volume
σ_m = constant (Appendix 1)
σ_n = volume of irreversibly fixed particles per unit filter volume (Delachambre)
σ_M = maximum value of σ
τ = retention age
τ_1, τ_2, τ_3 = particular values of τ
φ = implicit function
Ⓗ = absolute temperature (°K)
Φ = factor characterizing physico-chemical properties
Ω = bed area
Index f = outlet layer
Index i = inlet layer
Index o = clean filter bed

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